Euclidean Hyperspace and Its Physical Significance.

P. D. PEŠIĆ
St. John's College - Santa Fe, NM 87501-4599, USA

(ricevuto il 12 Febbraio 1993; approvato il 16 Giugno 1993)

Summary. — Contemporary approaches to quantum field theory and gravitation often use a four-dimensional space-time manifold of Euclidean signature (which we call «hyperspace») as a continuation of the Lorentzian metric. To investigate what physical sense this might have we review the history of Euclidean techniques in classical mechanics and quantum theory. Schwinger's Euclidean postulate gives a fundamental significance to such techniques and leads to a clearer understanding of the TCP theorem in terms of space-time uniformity. This is closely related to the principle of identicality that characterizes non-relativistic quantum theory. In quantum gravity we suggest that the Hartle-Hawking treatment of the wave function of the universe rests on a notion of space-time uniformity which can be related to the Euclidean postulate as a kind of «perfect cosmological principle» on the unobservable wave function of the universe which eliminates any *a priori* asymmetry between space and time. Euclidean hyperspace may mediate between the infinite-dimensional Hilbert space of quantum theory (whose metric is Euclidean) and the four-dimensional Lorentzian space-time of physical observations.

PACS 03.65 – Quantum theory; quantum mechanics.
PACS 03.70 – Theory of quantized fields.
PACS 04.60 – Quantum theory of gravitation.
PACS 02.90 – Other topics in mathematical methods in physics.

1. – Introduction.

Twentieth century physics has been largely concerned with a four-dimensional space-time characterized by an indefinite Lorentzian metric of signature $(-,+,+,+)$. However, much recent theoretical work in quantum field theory and quantum gravitation defines $\tau = it$ to continue to a Euclidean «hyperspace» (as we will call it) of signature $(+,+,+,+)$ in order to simplify calculations. There has been a tacit assumption that such methods are purely mathematical contrivances which are devoid of any physical significance. We will examine some ways in which Euclidean hyperspace may have important *physical* significance beyond its mathematical utility.

After surveying the history of the $O(4)$ symmetry of the Coulomb problem and the use of imaginary time variables in quantum-statistical mechanics and field theory, we will examine the Euclidean postulate which Schwinger introduced as a fundamental axiom of his source theory and which clarifies the significance of the TCP theorem [1]. We will argue that this postulate is closely related to a notion of indistinguishability and equality which we have elsewhere called the «principle of identicality,» raising the question of whether this principle should indeed be considered a fundamental postulate of quantum theory [2].

We then investigate what follows if such a principle of identicality is extended from timelike intervals to completely general intervals including spacelike ones. The motivation for this is that the identicality of particles should not only obtain for intervals immediately accesible to observation but also for those which will eventually become observable. This means that we require that the conditions of identicality must be uniformly extended from the interior of the light cone to the whole domain of space-time. Consequently, the principle of identicality takes the form of a hypothesis of uniformity of the fundamental states throughout space-time, which is also the purport of Schwinger's Euclidean postulate. Just as Einstein's theory denied that any «prior geometry» could prejudice the dynamical interrelation between matter and gravity, the Euclidean postulate denies any *a priori* asymmetry between space and time in the wave function of the universe. In so doing, we suggest that this postulate acts as a kind of «perfect cosmological principle» for quantum gravitation. We speculate also that Euclidean hyperspace forms the connection between the infinite-dimensional Hilbert space of quantum theory and the four-dimensional space-time of observable, relativistic, physics. As such, it may help connect the realms of the quantum and of gravitation.

2. – Historical development of Euclidean hyperspace.

The group of rotational symmetries in the Euclidean hyperspace, $O(4)$, was used even before the beginnings of the new quantum theory. $O(4)$ is the dynamical symmetry of the Schrödinger equation for the Coulomb potential, but it was known as an additional «hidden» symmetry for the Coulomb problem even in classical mechanics in the form of the so-called Runge-Lenz vector; Goldstein has reviewed its earlier history through the identification of this vector as an invariant of inverse-square forces by Laplace in 1799 and even earlier by Jakob Hermann in 1710 [3]. In fact, even before Schrödinger solved the hydrogen energy spectrum through his wave equation, Pauli independently solved this problem through the application of the Runge-Lenz vector [4]. Schwinger formulated this approach through the elegant technique of four-dimensional angular momenta in Euclidean hyperspace [5]. The more general implications of this symmetry are indicated by the circumstance that the Coulomb potential seems to have a special relation to the dimensionality of space; only an inverse-square law force will cancel the growth of surface area as the square of the radius, as is required if Gauss's theorem is to yield a conserved charge. It must be noted that this symmetry only obtains if the hydrogen atom is in a bound state. If it is unbound, the Lorentz group applies, as it does also if the full Dirac equation is used. Even in this simple example, the hidden group of Euclidean four-dimensional symmetry makes itself known in the approximate form of the non-relativistic equation. What is surprising is that in Schwinger's treatment the Euclidean group returns even in the context of the fully relativistic theory. With this in mind, the insight of the hidden $O(4)$ symmetry of hydrogen seems a far more significant and less isolated chapter in the development of quantum theory [6].

Other approaches further revealed the advantages of Euclidean techniques. Soon after the publication of Einstein's 1916 theory, Flamm embedded the spatial part of the Schwarzschild solution in a four-dimensional Euclidean manifold [7]. Bloch used an imaginary time variable in 1932 as a way of introducing finite-temperature Green's functions in many-body theory [8]. Wick used the Euclidean technique to simplify the Bethe-Salpeter equation [9] and Schwinger particularly emphasized the wide-ranging significance of the Euclidean domain for quantum field theory [10]. Such techniques are widely used in many areas of modern field theory as well as quantum gravitation. Hawking has gone as far as to assert that «we all know that quantum gravity has to be formulated in the Euclidean domain» [11]. This is an assertion that has been questioned both on technical and physical grounds. Yet such techniques are widely used at present, despite questions that have been raised concerning the propriety of «euclideanization» [12-16]. The question remains: are there *physical* grounds that would support the use of such a Euclidean principle as an axiom of quantum gravitation and field theory?

3. – The wave function of the universe and the Euclidean postulate.

As an example of the use of Euclidean techniques, let us turn to the approach of, Hartle and Hawking, in which the wave function $\Psi[h_{ij}]$ of a spatially closed universe with three-metric h_{ij} is defined through a Euclidean functional-integral prescription. They then use this Euclidean prescription to show that the wave function obeys the Wheeler-DeWitt equation with the «no boundary» condition [17]. Precisely because of the form of this boundary condition, the analyticity of Ψ is such that the Euclidean rotation of integration contours can be performed. Because of this analyticity, then, the Euclidean technique can be used even though the Wheeler-DeWitt equation is «hyperbolic» on a superspace whose metric has signature $(-, +, +, +, +, +)$. Hartle and Hawking infer, then, that the Euclidean functional-integral prescription «implies not only the Wheeler-DeWitt equation but also the boundary conditions which determine the ground-state solution» as well as permitting «the direct and explicit calculation of the semi-classical approximation».

Such semi-classical approximations lead to simple differential equations for the wave function of the universe reminiscent of the quantum theory of the hydrogen atom. For example, Hartle and Hawking note that the amplitude for a zero-volume three-sphere (in their minisuperspace model) is finite and non-zero, in contrast to the singularity evidenced by the classical theory for that case, which represents collapse of the universe to a point. They compare this to the familiar fact that in the s-state of the hydrogen atom the wave function does not vanish even at the origin, where presumably the electron and proton would coincide and where classical theory expects a singularity. We would only add to this analogy that the Euclidean symmetry of the wave function of the universe may be the analogue on the cosmic scale of the $O(4)$ «hidden symmetry» of the hydrogen atom, now applied to the whole universe considered as a bound state. But this playful analogy really calls for a more adequate account of why such Euclidean symmetries could ever apply either to the hydrogen atom or to the universe.

Our argument relies on the Euclidean postulate that Schwinger has formalized in his phenomenological theory of sources but which might be applied to quantum gravity as well. This postulate is best approached by reconsidering the TCP theorem.

4. – The TCP theorem and Schwinger's Euclidean principle.

The TCP theorem requires that if any physical process is possible, there must also be possible the same process with time reversed and parity inverted (TP) as well as particles changed to antiparticles (C). Basically, going from particle to antiparticle is equivalent to going to a particle reversed in space and time coordinates but with equal rest mass.

The essential difficulty this poses is that such a reversal risks violating causality, since the given process would be in the forward light cone (future), while the time-reversed process would be in the backward light cone (past). No Lorentz transformation corresponding to propagation less than the speed of light can link these two without possibly interchanging a cause with an effect. Then how can a theory based on relativity be consistent with this connection?

This difficulty is resolved in the formal proofs of the TCP theorem through the use of the so-called complex Lorentz group [18]. The ordinary group of Lorentz transformations Λ^{μ}_{ν} falls apart into four disconnected components characterized by the value of the determinant $\det\Lambda = \pm 1$ and by the sign $\Lambda^{0}_{0} = \pm 1$. This follows because of the invariant distinction between past and future which obtains in all such transformations and which characterizes each of these components. To apply the TP operator means going from one of these disconnected components of the group to another, since past and future are interchanged thereby. Since the complex group has only two disconnected components given by $\det\Lambda = \pm 1$, it can connect the past and future states characterized by sign $\Lambda = \pm 1$. The space-time variables are now allowed to be complex and hence the Lorentzian metric can be analytically continued to a Euclidean form in which the characteristic minus sign no longer distinguishes time from space variables.

So in order to «sneak» from the particle to the antiparticle we must go into a Euclidean space in which there is no causal distinction between past and future. But in order that causality be preserved, we must always re-emerge into Lorentzian space-time where physical processes are manifest and observable. This is the core of Schwinger's insight that the TCP theorem means that, attached to the Lorentzian space-time, there is a four-dimensional Euclidean manifold in which there is no essential difference between «space» or «time» coordinates.

Schwinger has formalized this attached Euclidean space in his theory of sources. He introduces sources which create or annihilate one particle (say) and insists that an overall source which creates and then annihilates a particle cannot be distinguished into component parts, «aside from reference to the space-time region that [a part]occupies» [19]. Schwinger notes that already creation and annihilation have been knit together inextricably so that nothing can distinguish the «creative» part of the source from the «annihilative» part, which he calls «the hypothesis of space-time uniformity». The continued identicality of particles as *indistinguishability* through creation and annihilation is implicit in his treatment. In defining the amplitude for the persistence of the vacuum state upon such single-particle creation and annihilation, Schwinger introduces a causal Green's function $\Delta_{+}(x-x')$ which appears to refer explicitly only to timelike separations of x and x'. But he insists that the spacelike region of $(x-x')$, where causality has no invariant meaning, must not lead to different values than those implied by the timelike region. Thus Schwinger considers that his hypothesis of space-time uniformity forbids «the existence of special relationships between sources» such as would follow from the addition of functions dependent upon the sign of the interval. To give this hypothesis «more precise, if rather abstract, form», Schwinger considers the four-dimensional Euclidean space attached to Minkowski space through the complex transformation $x_4 = ix^0$. He continues:

«There is no analogue in Euclidean space to the Minkowski distinction between timelike and spacelike intervals. Accordingly, special space-time structures would be rejected if one insisted that the invariant vacuum amplitude that describes a complete physical process continue to be meaningful and invariant on mapping the Minkowski space onto the Euclidean space. This is the Euclidean postulate [20].».

Schwinger then generalizes the source function to include charged sources. He shows that the sources for creation and annihiliation of charged particles can be chosen to be complex conjugate functions. These creation and annihilation functions cannot correspond to states of different mass. If they did, the Green's function $\Delta_+(x-x')$ would not have a unique extrapolation into spacelike regions since it would have to take into account the possibility of source functions corresponding to different masses of particles in different space-time regions. Here identicality as the *equality* of the masses of particle and antiparticle is directly related to the Euclidean postulate. Schwinger also shows that the creation of a particle at x' followed by its annihilation at x implies also the possibility of the creation of a possibly different complex-conjugated particle at x which then is annihilated at x'. This follows from the hypothesis of space-time uniformity since the «source» and «sink» aspects of the source cannot be separated and must correspond to each other exactly. Schwinger concludes:

«It is the principle of space-time uniformity that demands equal masses for the two kinds of particle, which are identified as particle and antiparticle. The Euclidean postulate produces the same conclusion through the absence of an invariant distinction between $x_4 - x_4' < 0$ and $x_4 - x_4' > 0$, which permits only one mass-parameter to appear [21].».

So the equality of the mass of particle and antiparticle is essential to the TCP theorem and it is easy to make the connection explicit. As Schwinger observes:

«The wider invariance introduced by the Euclidean postulate thus enables one to perform some discontinuous Lorentz transformations through the intermediary of continuous Euclidean transformations.».

Since $\Delta_+(x-x') = \Delta_+(-x+x')$ from the Euclidean postulate, the structure of all amplitudes is causally reversed under the TP transformation $x_\mu \to -x_\mu$. The C transformation then re-inverts the causal order, so that TCP restores the amplitude to its original form.

Although many treatments of the TCP theorem emphasize its grounding in the causality and locality of the relativistic theory, the theorem rests on a more general basis. Even a non-local theory which violates causality and the normal relation between spin and statistics can obey the TCP theorem as long as it obeys a weakened form of locality [22]. What seems to be indispensable is the *global* invariance of the theory as expressed by the Euclidean postulate.

5. – The Euclidean postulate and the principle of identicality.

What physical sense does this seemingly unphysical Euclidean hyperspace make? We suggest that it is intimately related to the principle of identicality. We have defined this principle in non-relativistic theory as the postulate that all observable physical states must be classes of exactly identical particles, in both the senses that they must be *equal* and *indistinguishable* [2]. Quantized fields indeed lead naturally to identical particles in the timelike region but such identical particles must remain identical whether they are in timelike or spacelike regions. However, no principle based on the ordinary Lorentz group can do this; Euclidean hyperspace permits global symmetries that will enforce identicality.

In order to establish the existence of a distinct species of identical particles, there must be a specific «marker» which characterizes each species of identical particles. Such a marker would need to be discrete so that individuals could not be distinguished within the species through continuous variation of the marker. For example, all electrons are characterized by their discrete and identical electric charge as well as by their spin and lepton number. In general, internal symmetries are necessary to give markers that will give observable identicality. Symmetries in space-time can only give a kinematic description which does not fully characterize identicality. Particles can exist virtually at other mass energies besides that necessary for real (on mass shell) existence, but their internal quantum numbers always remain the same and they establish the identity of the species of particle.

A simple connection of internal symmetries to the Lorentz group may be given by the Euclidean postulate. For instance, Schwinger has shown that the Euclidean postulate requires that every half-integral spin particle possesses a chargelike attribute [23]. In order to show this, Schwinger interprets the postulate to mean that «the Euclidean transcription may contain no indication of the original Minkowski space». This occurs naturally in his treatment of integer spins. However, in the case of half-integral spin, the time axis would be still singled out in Euclidean space unless one introduced an independent antisymmetric matrix of the form $q = \sigma_y$. This matrix then can be interpreted as a charge matrix. A heuristic way of seeing this is to note that all spinor fields have intrinsically two-valued representations [24]. These will be confused when the space-time reversal TP is made unless there is a distinct quantum number to differentiate the alternative C states; that is, there must be a charge quantum number to differentiate particle from antiparticle. In contrast, a particle of integer spin can be its own antiparticle and, in that case, need not consistently bear a charge. In this example the internal gauge symmetry corresponding to the conserved charge is closely related to the connection between the Minkowski and Euclidean spaces.

In order for there to be experimental observations of real quantities, the group chosen must be capable of yielding such quantities. It would seem as if this could be achieved in a real Euclidean hyperspace, but this turns out not to be the case, as a result of a general theorem of Cartan [25]. We require that real observable quantities should result from the underlying theory. Cartan addressed the question of such a «real image» emerging from the Euclidean space and showed that it is not possible to obtain a real image unless we pass to Lorentzian space-time. His demonstration begins by considering the general product of two identical spinors. This product can then be represented by what he calls a «bivector» with real components, an example of which is the electromagnetic field tensor $F_{\mu\nu}$, whose components are observable electric and magnetic fields. Such a bivector naturally emerges when the rotations depend on a specific parameter t in terms of which one then defines an «infinitesimal rotation» as occurring during dt [26]. However, such a «real image» of a physical bivector is only possible if one continues from Euclidean hyperspace to the pseudo-Euclidean space-time of special relativity. Interestingly, his argument requires that the spaces in question be of *even* dimensionality and may bear on the question of why space-time has four dimensions.

Cartan's argument is significant since it shows the way in which we may be forced to introduce the Minkowski metric on the underlying Euclidean space simply so that real fields are consequently observable in experiment. Further, it is the presence of a causal distinction between spacelike and timelike intervals which distinguishes Lorentzian from Euclidean spaces. The underlying Euclidean space still emphasizes the fundamental identicality of these two kinds of process in themselves. In this account, causality emerges as a concomitant of the *observability* of particles rather than as a fundamental quality of those particles in themselves. It is interesting that there appears to be no contradiction between the non-causal Euclidean space and the causal Lorentzian space-time. Rather, causality seems to be the condition for the observability of identical particles.

The Euclidean postulate may be suitable to be a fundamental postulate of quantum-gravitational theory since it enforces a «perfect cosmological principle» of space-time uniformity *on the level of the wave function, which is not directly observed.* In contrast, steady-state cosmology attempted to apply a perfect cosmological principle on the level of observable quantities; it was unable to do so without contradicting observation [27,28]. In the true spirit of quantum theory, the Euclidean postulate ensures that the fundamental level of cosmology, the wave function and its Hilbert space, is purged of any *a priori* asymmetry between space and time, in the same way that Einstein's theory prevents any «prior geometry» from infringing the dynamic relation between matter and gravitation [29].

6. – Concluding speculations.

Hawking has written that «one could take the attitude that quantum theory and indeed the whole of physics is really defined in the Euclidean region and that it is simply a consequence of our perception that we interpret it in the Lorentzian regime» [30]. Our argument has concerned what fundamental principle might indeed move one to adopt that attitude; the observed Lorentzian regime may emerge into our perception not out of arbitrary subjectivity but from the requirement that perception register only real numbers. Observable reality as it is manifest in Lorentzian space-time may be connected to another level of reality which is governed by Euclidean hyperspace. There, the symmetry between space and time is perfect and not yet abridged by the appearance of causal connections. Such a connection of Euclidean with Lorentzian geometry is reminiscent of the work of Klein which established the consistency of Euclidean and non-Euclidean geometries by showing the way in which each of these geometries can give consistent models of the others [31,32]. The geometries of Lobachevski and Riemann always depend on an observable parameter (the curvature of space) yet are consistently related to Euclidean geometry, which does not depend on any parameter. Similarly, the Lorentz group concomitant to observable phenomena may be consistently related to the Euclidean group. We speculate that this may be the significance of Schwinger's Euclidean postulate. One returns at this point to Minkowski's statement that «henceforth space by itself and time by itself are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality» [33]. Without denying this, we now might continue Minkowski's thought: space-time is the observable manifestation of an unobservable Euclidean hyperspace, in which the union of space and time is perfect.

Between the unobservable realm of the Hilbert space with its Euclidean metric and the observable realm of Lorentzian space-time there may be a middle term, the Euclidean hyperspace which mediates between the unobservable and the observable. In so doing it faces both towards the complex-valued realm of Hilbert space as well as towards the real-valued space-time of causal appearances.

* * *

I thank J. Schwinger for the conversation, many years ago, which led me to reflect on these matters. I also thank St. John's College and the Alfred P. Sloan Foundation for their support.

REFERENCES

[1] J. Schwinger: *Particles, Sources, and Fields* (Addison-Wesley, Reading, Mass., 1970 and 1973), 2 Vols.
[2] P. D. Pešić: *Am. J. Phys.*, **59**, 971, 975 (1991); a paper considering arguments for the fundamentality of the principle of identicality is in preparation.
[3] H. Goldstein: *Am. J. Phys.*, **43**, 737 (1975); **44**; 1123 (1976).
[4] W. Pauli: *Z. Phys.*, **36**, 336 (1926).
[5] See L. Schiff: *Quantum Mechanics* (McGraw-Hill, New York, N.Y., 1968), pp. 234-239.
[6] M. Bander and C. Itzykson: *Rev. Mod. Phys.*, 38, 330, 346 (1966).
[7] L. Flamm: *Phys. Z*, **17**, 448 (1916); see J. Eisenstaedt: in *Einstein and the History of General Relativity*, edited by D. Howard and J. Stachel (Birkhäuser, Boston, Mass., 1989), pp. 237-238.
[8] F. Bloch: *Z. Phys.*, **74**, 295 (1932).
[9] G. C. Wick: *Phys. Rev.*, **96**, 1124 (1954).
[10] J. Schwinger: *Proc. Nat. Acad. Sci. USA*, 44, 223, 956 (1958).
[11] S. W. Hawking: *Phys. Rev. D*, 37, 904 (1988).
[12] A. Anderson and B. DeWitt: in *Between Quantum and Cosmos*, edited by W. H. Zurek, A. van der Meerwe and W. A. Miller (Princeton University Press, Princeton, N.J., 1988), pp. 76-77.
[13] A. Folacci: *Phys. Rev. D*, **46**, 2553 (1992).
[14] P. O. Mazur and E. Mottola: *Nucl. Phys. B*, **341**, 187 (1990).
[15] G. F. R. Ellis: *Gen. Rel. Grav.*, **24**, 1047 (1992).
[16] T. Dereli and R. W. Tucker: *Class. Quantum Grav.*, **10**, 365 (1993).
[17] J. B. Hartle and S. W. Hawking: *Phys. Rev. D*, **28**, 2960 (1983).
[18] R. F. Streater and A. S. Wightman: *PCT, Spin and Statistics, and All That* (W. A. Benjamin, New York, N.Y., 1964), p. 9 and ff.; see also R. P. Feynman: *Elementary Particles and the Laws of Physics* (Cambridge University Press, Cambridge, 1987), p. 8 and ff.

[19] J. SCHWINGER: *Particles, Sources, and Fields*, (Addison-Wesley, Reading, Mass., 1970 and 1973), Vol. 1, p. 42.
[20] J. SCHWINGER: *Particles, Sources, and Fields*, (Addison-Wesley, Reading, Mass., 1970 and 1973), Vol. 1, p. 44; note that Schwinger uses the term «Minkowskian» for the space-time which we will call «Lorentzian», following what seems to be the more prevalent usage.
[21] J. SCHWINGER: *Particles, Sources, and Fields*, (Addison-Wesley, Reading, Mass., 1970 and 1973), Vol. 1, p. 47.
[22] P. ROMAN: *Introduction to Quantum Field Theory* (Wiley, New York, N.Y., 1969), p. 382-387.
[23] J. SCHWINGER: *Particles, Sources, and Fields*, (Addison-Wesley, Reading, Mass., 1970 and 1973), Vol. 1, pp. 111-113, 139-140.
[24] E. CARTAN: *The Theory of Spinors* (Dover, New York, N.Y., 1961), pp. 114-116.
[25] E. CARTAN: *The Theory of Spinors* (Dover, New York, N.Y., 1961), pp. 130-132.
[26] For illuminating discussions concerning bivectors and their use in physics see D. HESTENES: *Spacetime Algebra* (Gordon and Breach, New York, N.Y., 1966); D. HESTENES: *Am. J. Phys.*, **39**, 1013 (1971); W. E. BAYLIS, J. HUSCHLIT and JIANSU WEI: *Am. J. Phys.*, **60**, 788 (1992).
[27] H. BONDI: *Cosmology* (Cambridge University Press, Cambridge, 1988), pp. 11-15, 65-74.
[28] C. W. MISNER, K. S. THORNE and J. A. WHEELER: *Gravitation* (W. H. Freeman, San Francisco, Cal., 1973), p. 745.
[29] For the notion of «prior geometry» and its critique see C. W. MISNER, K. S. THORNE and J. A. WHEELER: *Gravitation* (W. H. Freeman, San Francisco, Cal., 1973), pp. 429-431.
[30] S. W. HAWKING: in *Recent Developments in Gravitation*, edited by M. LEVY and S. DESER (Plenum, New York, N.Y., 1979), pp. 146-147.
[31] R. BONOLA: *Non-Euclidean Geometry* (Dover, New York, N.Y., 1955), p. 164.
[32] M. J. GREENBERG: *Euclidean and Non-Euclidean Geometries* (W. H. Freeman, New York, N.Y., 1980), pp. 177 and ff., 237 and ff.
[33] See H. MINKOWKI in H. A. LORENTZ, A. EINSTEIN, H. MINKOWSKI and H. WEYL: *The Principle of Relativity* (Dover, New York, N.Y., 1923), p. 75.