Euler's Musical Mathematics

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Dedicated to Barry Mazur

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Of Leonhard Euler's thirty thousand published pages, only a few hundred are devoted to music, but these have special significance in his vast oeuvre, even though they are among his least-known works. Music was among the first topics he addressed at length, and he returned to it several times throughout his life. Moreover, musical questions led Euler to consider new mathematical topics and devise new approaches that then characterized several of his most important initiatives in mathematics and physics. Indeed, Euler's individual mathematical discoveries, great as they are, need to be placed in context of his larger role in the beginnings of modern number theory and topology. As familiar as these mathematical disciplines have become, we cannot take them for granted but should try to understand how they came into being in Euler's hands. In this story, his musical writings open surprising perspectives.

Euler stands in a long line of musical mathematicians, arguably reaching back to the Pythagoreans, who connected consonant musical intervals with simple ratios, such as the octave (2:1) and the perfect fifth (3:2). From Plato until the seventeenth century, music was studied as part of a "four-fold way" (*quadrivium*), alongside arithmetic, geometry, and astronomy. For Johannes Kepler, music was central to his search for planetary laws of motion in his *Harmonices mundi* (1619).[1] René Descartes's first work was a short *Compendium musicae* (1618); in subsequent years, he continued to correspond with Marin Mersenne on musical matters alongside questions in mathematics and physics. Mersenne himself considered music the central science, which he explored in his encyclopedic *Harmonie universelle* (1637).[2] Isaac Newton's youthful notes show his interest in musical ratios; he later tried to impose the musical octave on the color spectrum (1675).[3]

Early Musical Writings

Euler also began his studies early in his life, in a milieu that considered music a liberal art integrally connected with mathematics, not separate from it. At age 13 (1720), Euler matriculated at the University of Basel, which included musical studies in its curriculum and was an important center of musical thought. His father, a Calvinist pastor, introduced him to Johann Bernoulli (1667–1748), whom Euler visited on Saturday afternoons to discuss mathematics.

[1]See Peter Pesic, "Earthly Music and Cosmic Harmony: Johannes Kepler's Interest in Practical Music, Especially Orlando Di Lasso," *Journal of Seventeenth-Century Music* 11(1), (2005), http://www.sscm-jscm.org/v11/no1/pesic.html.

[2]For a fuller discussion of Descartes, Mersenne, Kepler, and Newton, as well as of Euler, Helmholtz, Riemann, and others, see Peter Pesic, *Music and the Making of Modern Science* (MIT Press, forthcoming, 2014).

[3]See ibid., chap. 8, and Peter Pesic, "Isaac Newton and the Mystery of the Major Sixth: A Transcription of His Manuscript 'Of Musick' with Commentary," *Interdisciplinary Science Reviews* 31 (2006), 291–306.

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Johann noted his extraordinary talents and persuaded Euler's father to allow his son to follow his mathematical interests; thereafter, Johann continued to correspond with Euler about mathematical, scientific, and musical questions, as did his son Johann II (1710–1790).

Indeed, Euler was much occupied with music throughout his life. Nicholas Fuss, his student, son-in-law, and secretary, recorded that "Euler's chief relaxation was music, but even here his mathematical spirit was active. Yielding to the pleasant sensation of consonance, he immersed himself in the search for its cause and during musical performances would calculate the proportion of tones."[4] This quest for a new mathematics of music persisted throughout his productive life.

Euler's earliest scientific notebooks include an outline he prepared at age 19 (1726) for a projected work he entitled "Theoretical Systems of Music," an ambitious survey for which he intended to include sections on composition in one and many voices, treating both melodic and harmonic writing.[5] His outline also envisaged chapters on various dances, as well as larger musical forms. Clearly, Euler's interest in music encompassed many aspects of contemporary composition and practical music making, not only its mathematical elements. The connections we will consider between music and mathematics should not be understood as "interdisciplinary," because Euler's early studies considered music and mathematics as part of a single coordinated whole, as the *quadrivium* had long mandated.

Indeed, in his early manuscripts, notes on musical theory precede any material relating to his second printed work, "Physical Dissertation on Sound" (1726), indicating the path that led him, already in his late teens, from music to the mathematical physics of sound.[6] Starting with the work of Newton and Johann Bernoulli the elder, Euler extended the mechanics of sound waves to wind instruments, an application of particular interest to him. Although beyond the scope of this article, Euler's early work on sound laid the foundation for his advocacy of the continuum cosmology, for his seminal work on fluid mechanics, as well as for his interest in the analogy between sound and light that led him to argue for a wave theory of light.[7]

During this same period, Euler was also working on a more speculative, larger work, his *Tentamen novae theorae musicae ex certissimis harmoniae principiis dilucide expositae* (*Essay on a New Theory of Music Based on the Most Certain Principles of Harmony Clearly Expounded*).[8]

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[4]From Fuss's "Eulogy in Memory of Leonhard Euler," in A. P. Yushkevich, N. N. Bogolyubov, and G. K. Mikhaĭlov (eds.), *Euler and Modern Science*, Mathematical Association of America, Washington, D.C., 2007, 375.

[5]See S. S. Tserlyuk-Askadskaya, "Euler's Music-Theoretical Manuscripts and the Formation of His Conception of the Theory of Music," in *Euler and Modern Science*, 349–360, Yushkevich et al., *Euler and Modern Science*, 75. For reproductions of Euler's notebooks, see H. Bredekamp and W. Velminki (eds.), *Mathesis & Graphé: Leonhard Euler und die Entfaltung der Wissenssysteme*, Akademie Verlag, Berlin, 2010, 39–64.

[6]For the original text, see "Dissertatio physico de sono," E2, III.1.183-196. The original text of this and other works by Euler may also be found in Leonhard Euler, *Opera Omnia*, B. G. Teubner, Leipzig, 1911. For convenience, I will cite them by the standard Eneström number of each item, here E2, and its place in the *Opera omnia* by series, volume, and pages, here III.1.183-296. These works (along with helpful listings of translations and secondary literature) can be found at the online Euler Archive at http://www.math.dartmouth.edu/~euler/. Euler's first published paper, "Constructio linearum isochronarum in medio quocunque resistente," E1, II.6.1-3, concerned the brachistochrone problem, finding a curve along which a particle falls in the shortest time. See C. Edward Sandifer, *The Early Mathematics of Leonhard Euler*, Mathematical Association of America, Washington, D.C., 2007, 3–5.

[7]See Leonhard Euler and C. Truesdell, *Rational Fluid Mechanics, 1687–1765: Editor's Introduction to Vol. II, 12 of Euler's Works*, Orell Füssli, Zürich, 1954; C. Truesdell, *The Rational Mechanics of Flexible or Elastic Bodies, 1638–1788: Introduction to Leonhardi Euleri Opera Omnia Vol X et XI Seriei Secundae*, Orell Füssli, Zurich, 1960; G. K. Mikhaĭlov and L. I. Sedov, "The Foundations of Mechanics and Hydrodynamics in Euler's Works," Yushkevich et al., *Euler and Modern Science*, 167–181; Lokenath Debnath, *The Legacy of Leonhard Euler: A Tricentennial Tribute*, Imperial College Press, London, 2010, 297–336. For an overview of Euler's relation to physics, see Dieter Suisky, *Euler as Physicist*, Springer, Berlin, 2009; for further discussion of Euler's work on the theories of sound and light, see Pesic, *Music and the Making of Modern Science*, chap. 10.

[8]See Hermann Richard Busch, *Leonhard Eulers Beitrag zur Musiktheorie*, G. Bosse, Regensburg, 1970; E. V. Gertsman, "Euler and the History of a Certain Musical-Mathematical Idea," Yushkevich et al., *Euler and Modern Science*, 335–347.

Unable to find a job in his native city, in 1727 Euler moved from Basel to St. Petersburg, where he obtained the chair of natural philosophy in 1730, the year he completed writing his *Tentamen*. By devoting so much of his attention to this work during the crucial period in which he needed to establish himself in a permanent position, Euler showed how integral he considered music to be to mathematics and natural philosophy.

Euler began his *Tentamen* by reviewing his earlier work on the physical basis of sound. Dissatisfied with the traditional Pythagorean lore that simple ratios such as 1:2 (octave) are *more perfect* than complex ones such as 243:256 (semitone), Euler argued that they were *more pleasurable* and calculated the exact degrees of pleasure involved.[9] Euler's calculus of sentiment pioneered a new mathematics of aesthetics, a field that remains scarcely explored.[10] To connect perceived feeling with mathematical order, he stipulated that "two or more sounds are pleasing when the ratio, which exists between the numbers of vibrations produced at the same time, is understood; on the other hand, dissatisfaction is present when either no order is felt or that order which it seems to have is suddenly confused." To make this quantitative, "we graded this perceptive ability in certain degrees, which are of the greatest importance in music and also may be found to be of great value in other arts and sciences of which beauty is a part. Those degrees are arranged in accordance with the ease of perceiving the ratios, and all those ratios that can be perceived with equal facility are related to the same degree." This he calls their *degree of agreeableness* (*gradus suavitatis*), which might be translated as *sweetness*, *charm,* or *tunefulness*.[11]

The priority the ancients had given to the intervals and ratios themselves Euler now assigned to the perceiving human subject.[12] For the first degree of agreeableness he takes the unison, 1:1 (which some ancient sources refused to consider an interval at all); for the second, the octave, 1:2; the ratios 1:3 (twelfth) and 1:4 (double octave) both occupy the third degree, because "which of these last two is the more easily perceived is disputable." Euler illustrates his reasoning with a diagram (Fig. 1) showing "the pulses in the air as dots placed in a straight line. The distances between the dots correspond to the intervals of the pulses," which he takes as visualizing their degree of understandability and hence agreeableness.

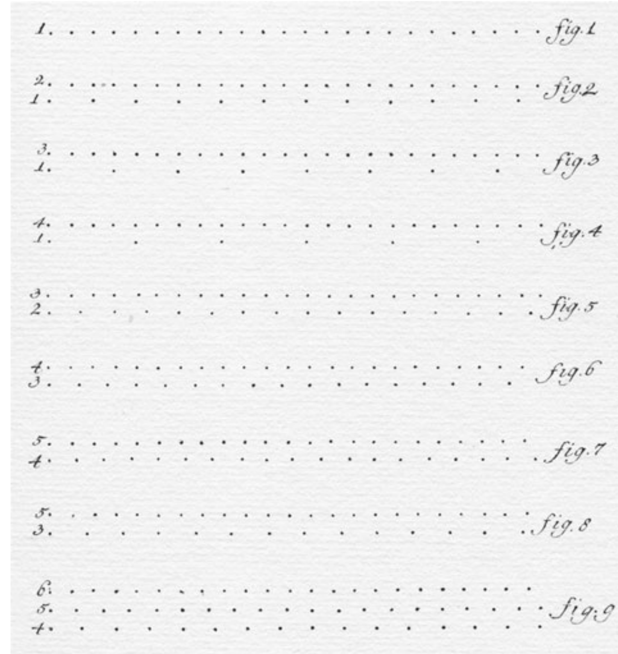

Figure 1. Euler's diagram visualizing the relative agreeableness of various simple ratios of sound pulsation, from his *Tentamen* (1739).

At the same time, though, this diagram represents the coincidences between the sound "pulses" and hence represents geometrically the interrelation between the sound waves. Implicitly, Euler's two different meanings converge: agreeableness correlated with the relative congruity of the two wave forms, which Hermann von Helmholtz made explicit in his physical theory of consonance more than a century later (with due acknowledgment to Euler).[13] Still, in his *Tentamen* Euler worked mostly within the older temperaments based on whole-number ratios, rather than the newer equal temperament, which requires division of the octave into 12 equal semitones, each given by the irrational factor $\sqrt[12]{2}$. For instance, J. S. Bach's *Wohltempierte Klavier* (1722) required a temperament capable of playing in all 24 major and minor keys.[14] As we shall see, Euler returned to this issue in later life.

With his chosen limitations, Euler's quest for a precise degree of agreeableness informed his mathematical

[9]In his earliest writings, Euler seems unaware of Leibnitz's 1712 comment that the beauty of music "consists only in the harmonies of numbers and in a calculation, which we do not perceive but which the soul nevertheless carries out, a calculation concerning the beats or vibrations of sounding bodies, which are encountered at certain intervals." See Walter Bühler, "Musikalische Skalen und Intervalle bei Leibniz unter Einbeziehung bisher nicht veröfftentlichter Texte I," *Studia Leibnitiana* 42 (2010), 129–161.

[10]Among the very few other attempts, note George David Birkhoff, *Aesthetic Measure*, Harvard University Press, Cambridge, Massachusetts, 1933. Birkhoff's basic equation, $M = \frac{O}{C}$ (where M is the aesthetic measure, O the order, and C the complexity), is consistent with Euler's approach.

[11]C. S. (Charles Samuel) Smith, *Leonhard Euler's Tentamen Novae Theoriae Musicae: A Translation and Commentary*, University Microfilms, Ann Arbor, 1974, 27–28. E33, III.1.197-427; Preface. All citations from this work will follow this translation, indicating also the chapter and section number.

[12]The traditional hierarchy of musical intervals simply assumed that "multiple" ratios, such as 1:n, and "superparticular" ratios, of the form (n + 1):n, were superior to other classes of ratios, without any further justification beyond their greater "simplicity."

[13]Hermann von Helmholtz, *On the Sensations of Tone as a Physiological Basis for the Theory of Music*, Alexander John Ellis (ed. and trans.), 2d English, Dover Publications, New York, 1954, 229–233.

[14]I thank Walter Bühler for pointing out to me that Euler discusses equal temperament in his early "Adversaria mathematica" (1726, f. 45r) and briefly in *Euler's Tentamen*, 204–205 (IX§17). Bach does not call for "equal" but "well" (presumably unequal) temperament, about whose detail there remains much controversy. For the continuing mathematical effects of earlier controversies about tuning, see Peter Pesic, "Hearing the Irrational: Music and the Development of the Modern Concept of Number," *Isis* 101 (2010), 501–530. See also Ross W. Duffin, *How Equal Temperament Ruined Harmony (and Why You Should Care)*, W. W. Norton, New York, 2007.

rankings. From his decision to assign the degree 1 to 1:1 and 2 to 1:2, Euler notes that "by the simple operation of halving or doubling, the degree of agreeableness is changed by unity." Accordingly, to ratios of the form $1{:}2^n$ he assigns the degree $(n + 1)$, because "the degrees progress equally in ease of perception. Thus, the fifth degree is perceived with more difficulty than the fourth," and so on. In light of this, he chooses the degree n always to be integral, never fractional "since in this case the ratio would be irrational and impossible to recognize," implying an underlying rationality to the felt quality of agreeableness. For ratios of the form $1{:}p$, where p is prime, he assigns the degree p, "by induction" (as he puts it), assigning both 1:3 and 1:4 to the same degree, namely 3. He then argues that $1{:}pq$ (where both p and q are prime) has degree $p + q - 1$. A few more steps lead him to the general conclusion that for any composite number m composed of n prime factors whose sum is s, the ratio $1{:}m$ has the degree of agreeableness $s - n + 1$. He concludes that the degree of a series of proportions such as $p{:}q$ or $p{:}q{:}r$ (where p, q, r are primes) is the same as of $1{:}pq$ or $1{:}pqr$, respectively, where Euler calls the least common multiple of these primes the *exponent* of the ratio.[15] Hence, he assigns to $1{:}pqr$ or $1{:}p{:}q{:}r$ the degree $p + q + r - 2$. Thus, the fifth (2:3) has degree $5 - 2 + 1 = 4$, the same as 1:6. He sets out the result in a table that goes far beyond the traditional set of musical ratios (Fig. 2).

Gr. II.	2:5.	*Gr. IIX.*	3:7.	3:64.	1:160.
1:2.	1:18.	1:14.	1:25.	1:256.	5:32.
Gr. III.	2:9.	2:7.	1:28.	*Gr. X.*	1:162.
1:3.	1:24.	1:30.	4:7.	1:42.	2:81.
1:4.	3:8.	2:15.	1:45.	3:14.	1:216.
Gr. IV.	1:32.	3:10.	5:9.	6:7.	8:27.
1:6.	*Gr. VII.*	5:6.	1:60.	1:50.	1:288.
2:3.	1:7.	1:40.	3:20.	2:25.	9:32.
1:8.	1:15.	5:8.	4:15.	1:56.	1:384.
Gr. V.	3:5.	1:54.	5:12.	7:8.	3:128.
1:5.	1:20.	2:27.	1:80.	1:90.	1:512.
1:9.	4:5.	1:72.	5:16.	2:45.	
1:12.	1:27.	8:9	1:81.	5:18.	
3:4.	1:36.	1:96.	1:108.	9:10.	
1:16.	4:9.	3:32.	4:27.	1:120.	
Gr. VI.	1:48.	1:128.	1:144.	3:40.	
1:10.	3:16.	*Gr. IX.*	9:16.	5:24.	
	1:64.	1:21.	1:192.	8:15.	

Figure 2. Euler's table of the first ten degrees of agreeableness of musical intervals.

Euler's mathematical schema leads him to include ratios that have no precedent in traditional music theory; the most important sixteenth-century theorist, Gioseffo Zarlino, had argued that only numbers up to 6 (the *senario*, as he called them) are allowable in musical ratios, but Euler makes a case for going beyond this limit. In so doing, and in the whole layout of his table of intervals, Euler makes consonance and dissonance really a matter of degree, as opposed to the traditional tendency to distinguish sharply between them. He is led to this notably innovative step by his mathematics, which phrases both in the same general language of ratios, as well as by his awareness of the expressive power of dissonance.

Euler thus found a new numerical index that, to some extent, correlates with traditional (and aural) judgments of relative consonance but is far more precise. Consider, for instance, a major triad formed in the ratios 4:5:6. As noted above, its degree will be the same as that of $1{:}4{\cdot}5{\cdot}6 = 1{:}120$, determined by the prime factors of its exponent, $60 = 2^2 \times 3 \times 5$, in which $s = 12$ and $n = 4$, so that the degree in both cases is $s - n + 1 = 9$. Euler's arguments explain, for example, why a major triad (such as C-E-G, with ratios 4:5:6) sounds "happier" than a minor triad (E-G-B, in ratio 5:6:7). In his scheme, the major triad is in the ninth, the minor in the fourteenth degree; the minor triad is therefore more "sad" because "joy is conveyed by those things which have a simpler, more easily perceptible order, and sadness is conveyed by those things whose order is more complex and more difficult to perceive."[16] Euler presented his species in compendious tables that visually juxtapose musical and mathematical notations (Fig. 3), showing how important he considered both and how he sought to bring them together.

Still, Euler's scheme has some disturbing features. As noted earlier, his approach assigns the same degree to an interval between *two* notes (in the example above, $1{:}pqr$) as to a *triad* (here, $p{:}q{:}r$), which seems in conflict with the more fundamental status of triads in the musical framework of conventional harmony. More troubling, Euler's scheme assigns the same degree to the most familiar (and "consonant") triadic harmony C–E–G as well as to a number of strong dissonances (such as 3:7 or 4:7), according to the older, qualitative listings of intervals. He later returned to the issue of including the previously proscribed number 7. But he never really addressed the fundamental problem that his system assigns the same degree to the dissonant major seventh chord C–E–G–B as it does to the consonant triad C–E–G.[17]

Music and Number Theory

To simplify calculations in his *Tentamen*, Euler was one of the first to apply logarithms to musical ratios.[18] This fairly obvious *musical* application then induces Euler to take a new *mathematical* step, because expressing a logarithm's magnitude calls for the use of irrational numbers in general. For example, Euler notes that "since the measure of the octave is log 2, which is 0.3010300 according to the table, and since the fifth is log 3 – log 2, or 0.1760913, the ratio of the octave to the fifth will be approximately 0.3010300/

[15]Note that both the sum and number of terms of $1{:}p{:}q{:}r$ are increased by 1 compared to $p{:}q{:}r$, so that the degrees $s - n + 1$ of both ratios are the same.

[16]Smith, *Euler's Tentamen*, 72 (II§14). For further discussion of the context and implications of the status of the minor mode, see Pesic, *Music and the Making of Modern Science*, chap. 9.

[17]As pointed out by James Jeans, *Science & Music*, Dover Publications, New York, 1968, 155–156, who uses the ratios 8:10:12:15 for the major seventh chord.

[18]Smith, *Euler's Tentamen*, 119–122 (IV§35–39). Euler seems unaware that he was anticipated in this by Bishop Juan Caramuel de Lobkowitz in 1670 and Christiaan Huygens in 1724, as well as by Leibniz; see Bühler, "Musikalische Skalen und Intervalle bei Leibniz," 159–161.

Figure 3. Euler's musical illustration of the first ten species of harmony, according to his degrees of agreeableness.

0.1760913." The advantage of using these musical logarithms challenged Euler to find workable approximations to their infinite decimal expressions: "In order to reduce this to smaller numbers, this ratio is changed into the following fraction:

$$1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2+\ddots}}}.$$

From this we can derive the simple ratios 2:1, 3:2, 5:3, 7:4, 12:7, 17:10, 29:17, 41:24, 53:31, of which the last is the closest to the true ratio."[19] These successive approximations come from truncating the fraction at successive points downward in the denominator of this continued fraction, a name John Wallis had coined only a few years previously (1695). Euler seems to have been the first to apply continued fractions to music, thereby reducing the irrational expressions of logarithms to a sequence of "simple ratios," in accordance with his musical starting point.[20]

In the years following the writing of the *Tentamen* (and as he prepared for its publication in 1739), Euler wrote "On continued fractions" (1737), the first sustained treatment of this new kind of mathematical object.[21] He realized that continued fractions, as they emerged in his musical treatment, provided an ideal means for expressing irrational numbers. In this paper, Euler presented the first proof that e is irrational by writing it as a continued fraction,

$$e=2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{1+\cfrac{1}{1+\ddots}}}}}}}}}}.$$

Euler often returned to continued fractions throughout his later work; although he applied them widely, he was first drawn to use them in addressing musical problems.[22]

Nor were the mathematical effects of his musical work restricted to this one particular technique. Though Euler's name later became so closely associated with number theory, his interest in this field began *after* his earliest work

[19] Smith, *Euler's Tentamen*, 121 (IV§38).

[20] Ibid., 16.

[21] Leonhard Euler, "An Essay on Continued Fractions," Myra F. Wyman and Bostwick F. Wyman (trans.), *Theory of Computing Systems* 18 (1985), 302–305. Original text E71, I.14.187-216.

[22] For his proof of the irrationality of *e*, see Euler, "An Essay on Continued Fractions." See also the discussion in Sandifer, *The Early Mathematics of Leonhard Euler*, 234–248; C. Edward Sandifer, *How Euler Did It*, Mathematical Association of America, Washington, DC, 2007, 185–190.

on music. In fact, the period of his greatest activity in number theory took place while he was preparing the *Tentamen*, so it was well after his arrival in St. Petersburg in 1727 and his subsequent correspondence with Christian Goldbach (who moved to Moscow shortly after Euler's arrival). Thus, in December 1729, Goldbach wrote Euler to ask him whether "Fermat's observation [is] known to you, that all numbers $2^{2^n}+1$ are prime? He [Fermat] said he could not prove it; nor has anyone else done so to my knowledge."[23] Euler's rather indifferent response indicates that, even by that date, he was not greatly interested in this fundamental question. Only after Goldbach prodded him in a subsequent letter did Euler catch fire; he then disproved Fermat's conjecture by showing that the fifth Fermat number, $2^{2^5}+1 = 4,294,967,297$, is divisible by 641.

After that, Euler read Fermat ever more closely and took up number theory with particular passion. His first result already underlines his phenomenal abilities as a calculator; such a factorization, without any mechanical aids, required great skill combined with mathematical acumen.[24] The same fascination with the pure manipulation and calculation of numbers also pervades his musical *Tentamen*, of which the tables shown previously are only a small sample of the many pages he devotes to lists of numbers connected with his musical scheme. Indeed, given Euler's ability to execute lightning mental computations of great complexity, one can readily imagine that he may have been able to compute degrees of agreeableness for what he was hearing, perhaps even in "real time." At the least, his *Tentamen* contains his retrospective account of musical awareness in terms of explicit arithmetic.

Even before he began his correspondence with Goldbach, Euler's absorption in the intricate arithmetic of his music theory provided fertile ground on which his ensuing interest in number theory could grow. The modern concept of "pure mathematics" should not blind us to the many ways in which, in Euler's time, no hard barrier separated it from the "applied" branches of what we now call physics, engineering, or music theory, all disciplinary names that he would not have known, much less separated absolutely. It was natural for Euler to follow his intricate musical arithmetic into the further studies of the properties of numbers that came to be called "number theory." According to André Weil, Euler's 1729 work was the "rebirth" of number theory, as Euler's work on the harmonic series and its generalizations marked "the birth of analytic number theory."[25]

Looking back to the *Tentamen*, many of Euler's musical arguments directly imply arithmetical problems that lead straight to the more general questions he later addressed about the properties of numbers. His definition $s - n + 1$ for the *gradus suavitatis* of a musical interval involves counting the n prime factors of the interval's exponent and their sum s; these became central topics in his ensuing number-theoretical work. The Pythagoreans had already investigated perfect numbers (each equal to the sum of its proper divisors, such as $6 = 1 + 2 + 3$) and pairs of amicable numbers, for which each is the sum of the other's proper divisors, such as 220 and 284. Both types of numbers became important to Euler, but he had already laid the groundwork for their study in his *Tentamen*. For any number m, Euler's $s(m)$ is the sum of its proper divisors. In a 1747 article, Euler further defined $\sigma(m)$ as the sum of *all* the divisors of m, including m itself, so that $\sigma(m) = s(m) + m$. Then two numbers j and m are amicable if $\sigma(m) = m + j = \sigma(j)$, a simple symmetric condition.[26] Euler also discovered 30 new pairs of amicable numbers, compared to the four known previously. His 1747 paper lists them in a format that is strikingly similar to his diagrams ranking musical intervals in the *Tentamen*.

Music and the Birth of the Topological Approach

The influence of Euler's musical work is also discernible in a very different arena of his activity, the new realm of mathematics that emerged with his famous solution to the problem of whether one could make a complete circuit of the Königsberg bridges (Fig. 4), returning to the starting point by crossing each of the seven bridges only once.

As late as 1736, Euler wrote that he considered this problem to be "banal," because its solution "bears little relationship to mathematics, and I do not understand why you expect a mathematician to produce it, rather than anyone else, for the solution is based on reason alone, and its discovery does not depend on any mathematical principle."[27] Later that same year, however, Euler must have changed his mind, for he now took what later would be called a "topological" approach to this problem as an example of a branch of geometry "that has been almost unknown up to now; Leibniz spoke of it first, calling it the 'geometry of position' [*geometria situs*]. This

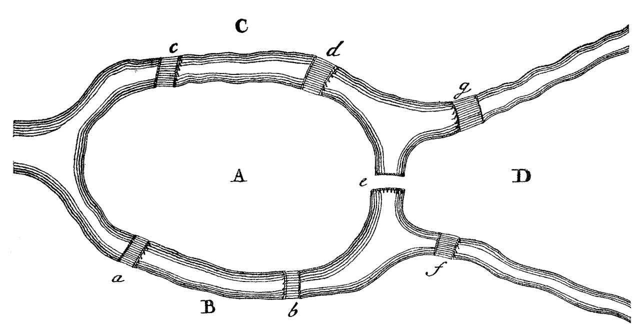

Figure 4. Euler's diagram of the city of Königsberg, the Kneiphof island (A), and the seven bridges over the River Pregel, *a, b, …, g*.

[23]Mark McKinzie, "Euler's Observations on Harmonic Progressions," in *Euler at 300: An Appreciation*, Robert E Bradley, Lawrence A. D'Antonio, and C. Edward Sandifer (eds.), Mathematical Association of America, Washington, D.C., 2007, 131–141. See also M. Bullynck, "Leonhard Eulers Wege zur Zahlentheorie," in Bredekamp and Velminski, *Mathesis & Graphé*, 157–175.

[24]I thank Noam Elkies for pointing out to me that 641 is the smallest natural candidate divisor of F_5; even so, demonstrating that it is indeed a divisor requires lengthy calculation.

[25]André Weil, *Number Theory: An Approach Through History from Hammurapi to Legendre*, Birkhäuser, Boston, 1984, 267, 3.

[26]William Dunham, *Euler: The Master of Us All*, Mathematical Association of America, Washington, D.C., 1999, 7–12. See E152, I.2.86-162, and also Sandifer, *How Euler Did It*, 49–62.

[27]Casper Hakfoort, *Optics in the Age of Euler: Conceptions of the Nature of Light, 1700-1795*, Cambridge University Press, Cambridge, 1995, 60–65, at 61.

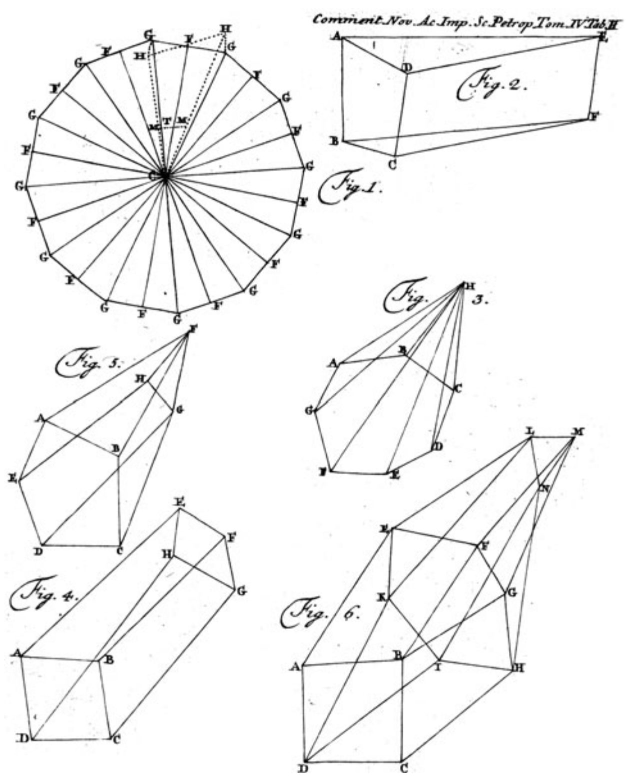

Figure 5. Euler's illustrations of polyhedra in his "Elements of the doctrines of solids" (1752).

branch of geometry deals with relations dependent on position alone, and investigates the properties of position; it does not take magnitudes into consideration, *nor does it involve calculation with quantities*."[28]

At that point, Euler generalized the Königsberg problem to "any configuration of the river and the branches into which it may divide, as well as any number of bridges, to determine whether or not it is possible to cross each bridge exactly once." Although Euler's 1736 paper is generally regarded as the origin of graph theory, that term was only introduced by J. J. Sylvester in 1878 and its terminology codified by George Pólya and others about 1936.[29] Euler reduced topography to alphabetic symbolism and derived simple rules, though without defining a numerical index that would "involve calculation with quantities," as he put it.

Euler later devised such an index when he returned to the "geometry of position" in his "Elements of the doctrines of solids" (1752), the first of two papers on the relations between the number of vertices (V), edges (E), and faces (F) of polyhedra (Fig. 5).[30]

Euler's crucial innovation here was to introduce the concept of the edge (*acies*) of a polyhedron, which, curiously enough, had never before been explicitly defined. Euler drew from Euclid the concept of a polyhedron's faces (*facies*) and its *angulus solidus*, here meaning not "solid angle" (in its present sense) but the point from which such an angle emerges, later called a "vertex" by Legendre (about 1794). If a solid polyhedron is bounded by plane faces, Euler concluded that "the sum of the number of solid angles plus the number of faces exceeds the number of edges by 2," or $V + F - E = 2$, "Euler's polyhedral formula." Here the requirement of closure for the polyhedron corresponds to the connectedness of an Euler walk in the Königsberg problem.[31] By identifying V, F, and E, Euler now could define the index $V + F - E = 2$.

The structure of this relation is strikingly similar to the degree of agreeableness of musical intervals. Both $V + F - E = 2$ and $s - n + 1 = d$ provide a general categorization of polyhedra and musical intervals, respectively, subsuming their individual differences under a larger genus, although Euler's musical degree was more general than his

[28]"The Seven Bridges of Königsberg," in J. R. Newman (ed.), *World of Mathematics*, Simon and Schuster, New York, 1956, 1:573–580 (emphasis added). Original text E53, I.7.1–10. See also B. Mahr and W. Velminski, "Denken in Modellen: Zur Lösung des Königsberger Brückenproblems," in Bredekamp and Velminski, *Mathesis & Graphé*, 85–100.

[29]See Norman Biggs, E. Keith Lloyd, and Robin J. Wilson, *Graph Theory, 1736–1936*, Clarendon Press, Oxford, 1986. See also W. Velminski (ed.), *Leonhard Euler, die Geburt der Graphentheorie: Ausgewählte Schriften von der Topologie zum Sodoku*, Kulturverlag Kadmos, Berlin, 2009.

[30]Elementa doctrina solida," E230, I.26.71–93; "Demonstratio nonnullarum insignium proprietatum, quibus solida hedris planis inclusa sunt praedita" E231, I.26.94–109. For commentary, see Sandifer, *How Euler Did It*, 9–18.

[31]Note that Euler states his conclusion verbally, rather than algebraically. For an excellent presentation of the details of both arguments and their connections, see David S. Richeson, *Euler's Gem: The Polyhedron Formula and the Birth of Topology*, Princeton University Press, Princeton, New Jersey, 2008. See also Debnath, *The Legacy of Leonhard Euler*, 153–173.

polyhedral formula, which only later was generalized to the "Euler characteristic" $\chi = V + F - E$. Indeed, there was scarcely any precedent before Euler for defining such an index where it did not obviously present itself. The degree of a polynomial equation is far more manifest in its algebraic expression than would be the putative definition of the "degree" of a polyhedron, much less of a musical interval, where it had no previous meaning. In his musical work, Euler first devised the general classificatory strategy he then applied to the polyhedron problem by defining a numerical index that would establish a clear taxonomy unifying all convex polyhedra.

Euler thus discovered not just the first important insights that later grew into the field of combinatorial topology, but, more deeply, discovered *indexing* as a crucial (and novel) tool of what became *the topological approach itself*. Music was a peculiarly appropriate first venue for this new topological thinking, because musical intervals do not have the kind of spatial structure that seems to govern elementary geometry. The lack of visible evidence—and his judgment of the insufficiency of the traditional criterion of "simplicity" of ratio—opened the door to his definition of degree, which he ultimately tied to his *auditory* criteria of *suavitas*. After Euler took this initial step away from the traditional givens of mathematics, such as pure ratios, it was much easier to think in essentially the same way when he came to the Königsberg problem and then to polyhedra. For each, Euler devised a degree that would have an invariant significance, bringing together particular cases previously considered quite distinct.[32]

Later Musical Writings

During the remainder of his long life, Euler returned to musical questions several times, reaffirming and reconsidering his youthful work in the *Tentamen*, especially the issue of intervals involving the number 7. In a series of papers beginning in 1760, Euler was among the first to argue that the number 7 was essential to the chord he and his contemporaries were beginning to call the dominant seventh (Fig. 6).[33]

Euler rightly notes the importance of this dissonant chord in the heightened expressivity of what he calls "modern," as opposed to "ancient," music. This continues and complements his account of musical "sadness," mentioned earlier. Euler also addressed the issue of equal temperament, showing that he had become aware of its prevalence and musical importance, although he argued that "the ear is not bothered by this irrational proportion," because it can be approximated by whole-number ratios.[34] Thus he remained in this sense faithful to the Pythagorean vision of whole numbers as the true basis of music. Nonetheless, Euler praises "modern" music as "sublime, because its character consists in a higher degree of harmony," compared to ancient music as "common [*commune*]," in the sense of adhering to common harmonic practice.[35] Yet he never cites a single musical example that would give specific insight into his compositional tastes; the only composer he ever mentions is Rameau, but then only as a theorist. Disconcertingly, his sole extended musical

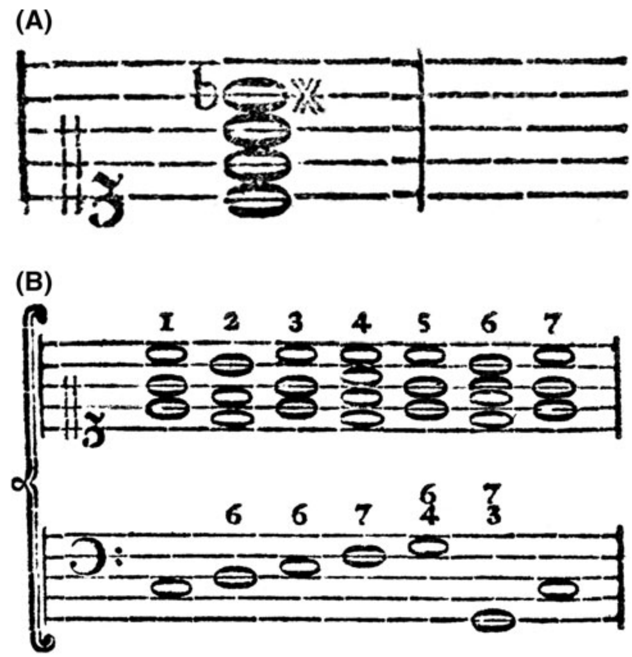

Figure 6. (A) Euler's example of a dominant seventh chord built on the note C (identified by the C clef on the bottom line of the staff): C, E, G, B♭, as dominant seventh in the key of F. (B) A progression cited by Euler outlining the key of C especially through the penultimate dominant seventh chord (marked $^{7}_{3}$). Note the parallel octaves between the lower voices, from the second to the third (D–E) and fourth to the fifth (F–G) chord. From "On the True Character of Modern Music" (1764).

[32]Modern music theorists have followed Euler's lead in exploring the geometry and topology of music. Martin Vogel, *On the Relations of Tone*, V. Kisselbach (trans.), Verlag für Systematische Musikwissenschaft, Bonn, 1993, 108, argues that Euler's 1773 work was a precursor of Arthur von Oettingen's 1866 concept of the *Tonnetz*, the generalized tonal pitch space ("tone net") taken up by the theorist Hugo Riemann, but already present in Euler's *Tentamen* according to Michael Kevin Mooney, "The 'Table of Relations' and Music Psychology in Hugo Riemann's Harmonic Theory" (Ph.D. diss., Columbia University, New York, 1996), 29–30. For a stimulating presentation of musical theory in relation to topology, see Dmitri Tymoczko, *A Geometry of Music: Harmony and Counterpoint in the Extended Common Practice*, Oxford University Press, New York, 2011.

[33]See his "Conjecture on the reason for some dissonances generally heard in music" (1760), E314, III.1.508–515; "On the True Character of Modern Music" (1764), E315, III.1.516–539; "On the True Principles of Harmony Represented in the Mirror of Music" (1773), E457, III.1.568–587, discussed further in Pesic, *Music and the Making of Modern Science*, chap. 10. Euler's priority in his analysis of the dominant seventh was noted in 1840 by François-Joseph Fétis, *History of Harmony*, Mary I. Arlin (trans.) Pendragon Press, 1994, 97, although in general Fétis is very critical of Euler's approach (see 69–84). See also Benjamin Downs, "Sensible Pleasure, Rational Perfection: Leonhard Euler and the German Rationalist Tradition," *Mosaic: Journal of Music Research* 2 (2012), http://mosaicjournal.org/index.php/mosaic/article/view/41/45.

[34]See his "Conjecture" (1760), E314, ¶¶7–16.

[35]See "On the True Character of Modern Music" (1764), E315.

example is a formulaic cadence that violates elementary rules of voice leading by allowing parallel octaves (Fig. 6B). Were these solecisms just typos, or did the great mathematician finally have a tin ear?[36] Or was he quoting crude hymnody he remembered from the Calvinist services of his childhood?

Perhaps our awe at Euler's seemingly superhuman abilities would have been tempered by hearing what really went on during his musical evenings. In any case, contemplating his musical preoccupations augments our sense of his humanity. Euler's serious and long-sustained engagement with music significantly affected his work and helped him open doors into new mathematical realms.

[36] I thank Noam Elkies for pointing out to me these problems in Euler's voice leading. Also thanks to Walter Bühler, Alexei Pesic, and Paul Espinosa (Curator, George Peabody Library Rare Books, Johns Hopkins University) for their generous help. Figures 4 and 5 appear courtesy of the George Peabody Library, The Sheridan Libraries, The Johns Hopkins University.