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Physics in Perspective

Helmholtz, Riemann, and the Sirens: Sound, Color, and the "Problem of Space"

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Emerging from music and the visual arts, questions about hearing and seeing deeply affected Hermann Helmholtz's and Bernhard Riemann's contributions to what became called the "problem of space [*Raumproblem*]," which in turn influenced Albert Einstein's approach to general relativity. Helmholtz's physiological investigations measured the time dependence of nerve conduction and mapped the three-dimensional manifold of color sensation. His concurrent studies on hearing illuminated musical evidence through experiments with mechanical sirens that connect audible with visible phenomena, especially how the concept of frequency unifies motion, velocity, and pitch. Riemann's critique of Helmholtz's work on hearing led Helmholtz to respond and study Riemann's then-unpublished lecture on the foundations of geometry. During 1862–1870, Helmholtz applied his findings on the manifolds of hearing and seeing to the *Raumproblem* by supporting the quadratic distance relation Riemann had assumed as his fundamental hypothesis about geometrical space. Helmholtz also drew a "close analogy … in all essential relations between the musical scale and space." These intersecting studies of hearing and seeing thus led to reconsideration and generalization of the very concept of "space," which Einstein shaped into the general manifold of relativistic space-time.

Key words: Hermann von Helmholtz; Bernhard Riemann; Albert Einstein; *Raumproblem*; space; color; music; general relativity; geometry; physiological optics; psychological acoustics; manifold.

> I admire ever more the original, free thinker Helm[holtz].
> Albert Einstein, letter to Mileva Marić, August 1899[1]

Introduction

The protean activities of Hermann Helmholtz resonated sympathetically with Albert Einstein, first expressed in this youthful love letter to his fiancée. Around 1903, Einstein and his friends in the "Olympia Academy" read Helmholtz as well as Bernhard Riemann, the visionary mathematician whose work on curved multidimensional manifolds was to prove central in the development of general relativity.[2] Both Einstein and Helmholtz were deeply interested in fundamental

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principles of science, such as the law of conservation of energy that Helmholtz advanced so powerfully and that Einstein inscribed in relativistic dynamics. Both were devoted to music; both were concerned with light, Helmholtz with its physiology, Einstein with its speed and interactions with matter. Both were engaged by what became called the "problem of space [*Raumproblem*]" that emerged in the wake of Riemann's work: what are the possible geometries of geometrical space and how do they relate to physical experience? Though hearing, seeing, and curved space may seem unrelated topics, they were connected in the activities of Riemann and Helmholtz and ultimately of Einstein, who used their ideas to shape the general theory of relativity.

Helmholtz's deep involvement in the nature of vision and hearing rested on his concerns with music and visual art, as well as on his medical and physiological interests. The dialogue between these arts and their respective sensory modalities fed strongly into his later investigations into the possible "spaces" of experience, in the sense of the multidimensional manifolds considered in Riemann's seminal 1854 lecture, "On the Hypotheses that Lie at the Basis of Geometry." Yet Helmholtz did not read this then-unpublished work until *after* responding to Riemann's posthumous, unfinished 1866 work on "The Mechanism of the Ear." This little-known chronology gives important context for Helmholtz's ensuing 1868 response, "On the Factual Foundations of Geometry," which relied on his studies of sound and color to set forth an empirical basis for Riemann's hypotheses. As we shall see, Einstein himself noted the direct line leading from Helmholtz's and Riemann's work on the sensory manifold to the curved, non-Euclidean manifold that is the centerpiece of the world geometry at the heart of general relativity.

Helmholtz's Investigations of Vision and Hearing

Helmholtz's extraordinary life trajectory, spanning activity and mastery in many fields, was already celebrated in his own lifetime, well before Einstein's birth. Though deeply interested in physics from early youth, family circumstances dictated Helmholtz's initial career as an army surgeon (1843–1848). Even while completing his onerous duties, he completed his seminal 1847 essay, "On the Conservation of Energy," which was of great importance in establishing the fundamental status of that principle.[3] In his ensuing activities as professor of physiology at Königsberg (1849–1855), Helmholtz (figure 1) undertook a characteristically exhaustive and extensive study of many aspects of nerve action, which began with innovative experimental studies. He succeeded in measuring the velocity of propagation of nerve impulses (1850), a feat others had doubted was even possible, given the great celerity of those impulses.[4] To accomplish this, Helmholtz had to invent a new myograph (figure 2). This led, later that year, to his general study of methods of measuring the extremely small time intervals involved in this new arena of experimental physiology, for which *time itself* became both an

Fig. 1. Hermann Helmholtz (1821-1894). *Source*: Helmholtz, *Wissenschaftliche Abhandlungen*. Erster Band (ref. 3), frontispiece.

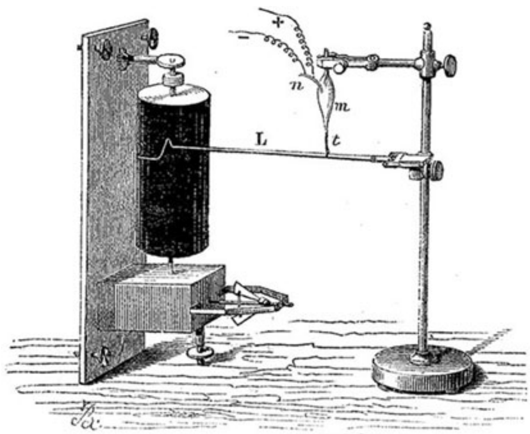

Fig. 2. Helmholtz's myograph, used to measure the time required for nerve conduction in the thigh muscle of a frog. *Source*: É.J. Marey, *La machine animale: locomotion terrestre et aérienne*. Quatriéme Édition (Paris: Ancienne Librairie Germer Bailliere et C^{ie}, 1886), p. 30.

experimental desideratum and an avenue to the attendant theoretical and philosophical questions, to which he and many others had been alerted by the work of Kant: were time and space deduced from experience or projected from the

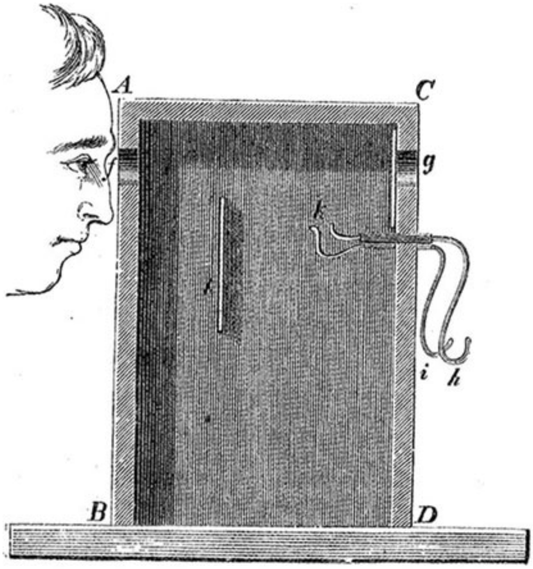

Fig. 3. Helmholtz's tachistoscope, used to avoid involuntary movement of the eye by very brief illumination of test images. *Source*: Helmholtz, *Handbuch der physiologischen Optik* (ref. 10), p. 567; *Treatise on Physiological Optics*. Vol. II (ref. 11), p. 197.

structure of the mind?[5] Thus, Helmholtz's tachistoscope (figure 3) was designed to obviate the extraneous effects of eye movement by illuminating the eye with an extremely short burst of light, giving a nearly instantaneous image of the eye's position.

The *annus mirabilis* 1850 also included Helmholtz's most famed optical invention, the ophthalmoscope, still in use today to examine the retina and the fundus of the eye.[6] But beside this well-known medical instrument, he also introduced many others, including the ophthalmotrope, a mechanical model to demonstrate eye movements (figure 4). Such devices helped him develop a "sign theory" that associated each such movement and its muscular state with the attendant visual perceptions, no longer considered as realities in themselves but as symbols of underlying physiological states and their external correlates.[7]

Here, as throughout his career, Helmholtz used his experimental findings to ground his theoretical work.[8] Thus, his work on the mechanisms of vision led to his 1855 paper, "On the Theory of Complex Colors," in which he revived the three-color hypothesis of Thomas Young and gave it new and fuller support from his own investigations.[9] Helmholtz's outpouring of specialized researches on many aspects of human vision finally led to his massive *Handbuch der physiologischen Optik* (*Handbook of Physiological Optics*), whose first edition appeared in three parts during 1856–1866, a *summa* whose synthetic breadth and systematic rigor put

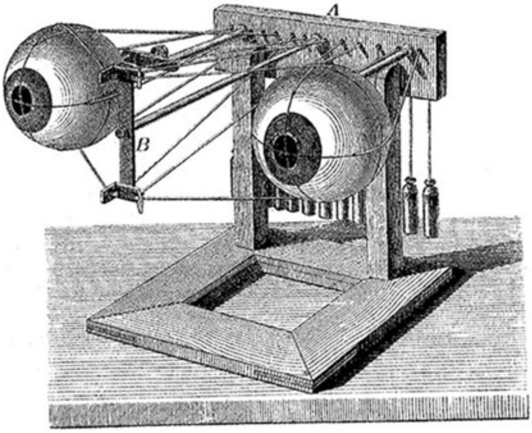

Fig. 4. Helmholtz's ophthalmotrope, a model used to study basic mechanisms of eye movements. *Source*: Helmholtz, *Handbuch der physiologischen Optik* (ref. 10), p. 326; *Treatise on Physiological Optics*. Vol. II (ref. 11), p. 197.

the entire field of physiological optics on a new plane of activity by applying physical principles to anatomical structures.

In part, Helmholtz accomplished this by including a historical dimension in his work, both to establish its sources and to make explicit its fundamental presuppositions. In the midst of his experimental studies, he was constantly looking to the larger theoretical questions he hoped to resolve, which historical awareness helped him formulate more pointedly. Thus, his awareness of Young's three-color hypothesis helped him formulate the relation between human physiology and the purely physical theory of color presented by Isaac Newton. Helmholtz also incorporated the 1853 work of Hermann Grassmann on color mixing to provide various geometrical representations of color perception (figure 5 *upper*), for which Helmholtz used the terms "curve" (*Curven*), "color circle" (*Farbenkreis*), "color cone" (*Farbenkegel*), or "color pyramid" (*Farbenpyramide*) in his 1867 *Handbuch*.[10] In this edition, he does *not* use the terms "manifold" or "space" (*Raum*), terms whose unfolding and gradually expanding meanings are crucial to the developments we will examine.

Helmholtz used these diagrammatic representations to clarify three independent parameters of perceived color, which today are called hue, saturation, and lightness, and which his work was extremely important in clarifying in the face of pervasive confusion about the exact meaning of these terms and the nuances between them.* In brief, the linear sequence of the Newtonian spectrum, arranged

* In present terminology, *hue* is the degree to which an area appears similar to the perceived colors red, yellow, green, blue or a combination of them; *saturation* its colorfulness relative to its own brightness; *lightness* (or *value*) its brightness relative to a similarly illuminated white.

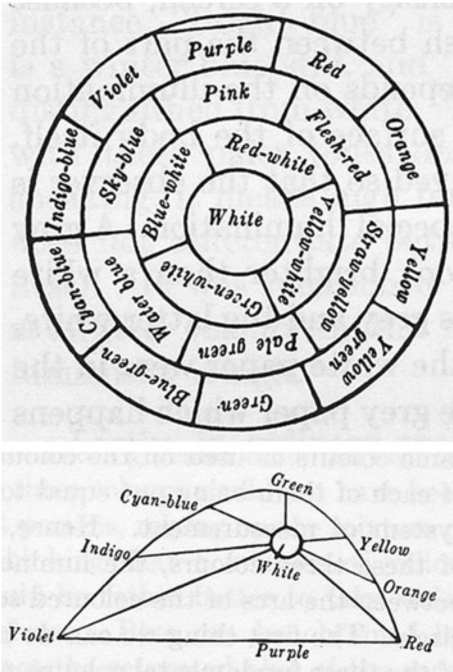

Fig. 5. (*upper*) Helmholtz's representations of Newtonian color theory using a "color circle" in which more saturated colors are near the circumference; this leaves out differences in luminosity. (*lower*) Helmholtz contrasts this with a markedly asymmetric curve showing the relation between colors of equal luminosity. *Source*: Helmholtz, *Handbuch der physiologischen Optik* (ref. 10), pp. 325, 332; *Treatise on Physiological Optics*. Vol. II (ref. 11), pp. 282, 288.

from red to violet, is perceived by the human eye in a decidedly nonlinear way. Helmholtz's diagram (figure 5 *lower*) shows that, to mix colored lights to form white, a different amount of yellow must be mixed with indigo, as compared with the relative amounts of orange and cyan-blue needed to produce white. In this diagram, these differences show up in the asymmetric shape of the overall curve, whose skew toward the red-orange side reflects the higher sensitivity of human daytime vision to those colors, as compared with the blue-violet side.

In the course of this work, Helmholtz also devoted attention to the possibility of describing the perceived distances between colors

> on the principle of the musical scale, because this seemed to be the best method for physiological reasons. Thus, colours whose wave-lengths are in the same ratio as the interval of a semi-tone between two musical notes are always at equal distances apart in the drawing [figure 6 *upper*]; or, to put it mathematically, equal distances in the drawing correspond to equal differences between the logarithms of the wave-lengths.[11]

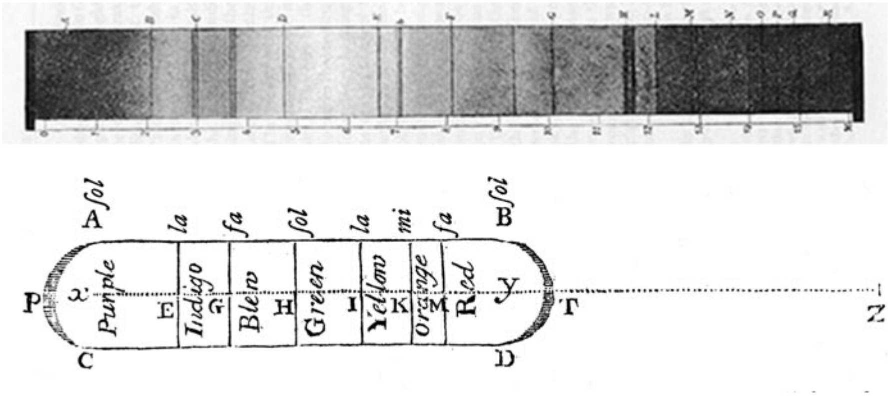

Fig. 6. (*upper*) Helmholtz's plate showing the solar spectrum with the more prominent Frauenhofer lines indicated in capital letters (above the corresponding dark lines) and a numerical scale (below) showing the correspondence between musical intervals of a semitone (labeled by successive numbers) and the spectral colors. The Frauenhofer C line roughly corresponds to red; E to green; F to "cyan-blue"; H–L to violet. *Source*: Helmholtz, *Handbuch der Physiologischen Optik*. Dritte Auflage. Zweiter Band (ref. 11), facing p. 54; *Treatise on Physiological Optics*. Vol. II (ref. 11), facing p. 64. (*lower*) Isaac Newton's comparison between the musical scale and the spectral colors. Newton introduced indigo and orange to fill out the analogy between a complete spectrum and the seven diatonic notes in an octave. *Source*: Is. Newton, "An Hypothesis explaining the Properties of Light, discoursed of in my several Papers" [1675-1676], in Thomas Birch, *The History of the Royal Society of London*. Vol. 3. A Facsimilie of the London Edition of 1756-57 (New York and London: Johnson Reprint Corporation, 1968), pp. 248-305, on p. 263.

Helmholtz approaches this parallelism in terms of Newton's imposition of the musical scale on the chromatic spectrum (figure 6 *lower*).

> The different sensations of colour in the eye depend on the frequency of the waves of light in the same way as sensations of pitch in the ear depend on the frequency of the waves of sound; and so, many attempts have been made to divide the intervals of colour in the spectrum on the same basis as that of the division of the musical scale, that is, into whole tones and semi-tones. Newton tried it first. However, at that time the undulatory theory was still undeveloped and not accepted; and not being aware of the connection between the width of the separate colours in the prismatic spectrum and the nature of the refracting substance, he divided the visible spectrum of a glass prism, that is, approximately the part comprised between the lines *B* and *H*, directly into seven intervals, of widths proportional to the intervals in the musical scale…; and so he distinguished seven corresponding principal colours; *red, orange, yellow, green, blue, indigo*, and *violet*.[12]

Helmholtz's spectral diagram (figure 6 *upper*) shows only about nine semitones (hence slightly more than a major sixth) between red (*B*) and violet (*H*), not the twelve semi-tones needed to span an octave between them, the overall interval Newton had assumed. In his diagram, Helmholtz's entire spectrum (*A*-*R*) spans sixteen semi-tones, almost an octave and a fourth, because his experimental work had shown that the ultraviolet wavelengths (*L*-*R*)

> are not invisible, although they certainly do affect the eye comparatively much less than the rays of the luminous middle part of the spectrum between the lines *B* and *H*. When these latter rays are completely excluded by suitable apparatus, the ultra-violet rays are visible without difficulty, clear to the end of the solar spectrum.[13]

Thus, his "scale of colours analogous to the notes of the piano," with yellow as middle C, extends from the "end of Red" as the F# below middle C to the highest visible ultraviolet frequency as the B above it.[14]

These investigations showed him that "this comparison between music and colour must be abandoned," both because "the spectrum is broken off arbitrarily at both ends," hence its divisions into colors are "more or less capricious and largely the result of a mere love of calling things by names."[15] Most of all, the eye's sensitivity varies greatly:

> [A]t both ends of the spectrum the colours do not change noticeably for several half-tone intervals, whereas in the middle of the spectrum the numerous transition colours of yellow into green are all comprised in the width of a single half-tone. This implies that in the middle of the spectrum the eye is much keener to distinguish vibration-frequencies than towards the ends of the spectrum; and that the magnitudes of the colour intervals are not at all like the gradations of musical pitch in being dependent on vibration-frequencies.[16]

As remarkable as visual perception may be, Helmholtz's critique brought forward important respects in which it falls short of the ear's capabilities to discriminate between musical pitches.

With this in mind, starting in 1852 and overlapping with his ongoing visual researches, Helmholtz began a no less sustained and exhaustive series of investigations into the physiology of hearing. This was close to his own personal inclinations, for he had played the piano since childhood, growing up in a musical household in a music-loving country and era.[17] When he went off to university (taking his piano with him), his father warned him not to allow his "taste for the solid inspiration of German and classical music be vitiated by the sparkle and dash of the new Italian extravagances…."[18] Of course, Helmholtz, as a true *Kulturträger* of his time, was also well acquainted with the masterworks of visual arts and from 1871 to 1873 gave a series of popular lectures "On the Relation of Optics to Painting."[19]

Helmholtz's investigations into music, sound, and hearing began during his Königsberg period and grew after he became the professor of anatomy and physiology at Bonn (1855–1858), where in 1856 he wrote "On Combination Tones," and then professor of physiology at Heidelberg (1858–1871), where in 1860 and 1862 he

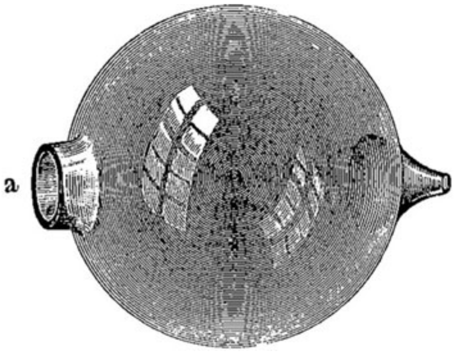

Fig. 7. Helmholtz's resonator to isolate an overtone. *Source*: Helmholtz, *Die Lehre von den Tonempfindungen*. Zweite Ausgabe (ref. 31), p. 74; *On the Sensations of Tone* (ref. 25), p. 43.

wrote the papers "On Musical Temperament" and "On the Arabic-Persian Scale."[20] These few samples show something of the breadth of his investigations, for his interest in music led him to explore beyond the confines of European practice in order to study the cultural determination of hearing. The entire project eventuated in his masterwork, *Die Lehre von den Tonempfindungen als physiologische Grundlage für die Theorie der Musik* (*On the Sensations of Tone as a Physiological Basis for the Theory of Music*), first published in 1863, whose title proclaims *music* as the true object of his study; in contrast, his *Handbook of Physiological Optics* makes no mention of painting or the visual arts.[21] His central term *Empfindung* connotes not only "sensation" but also "expression" in its full musical sense.

As with his studies of vision, Helmholtz developed or improved many instruments to undertake experimental examination of the issues that emerged, such as the glass resonators he used to isolate overtones and render them more audible (figure 7).[22] In this case, the resonator acted to amplify a sonic phenomenon so that it was more amenable to careful scrutiny. In other cases, Helmholtz devised means of translating and recording sonic events in a visual form, including their time dependence (figure 8). In this way, a tuning fork can be made to inscribe its sinusoidal vibrational pattern along a moving strip of paper, producing a visible trace that diagrammatically graphs space against time.

So far, Helmholtz's sonic investigations had stayed with the study of vibrating bodies, but he realized (following the earlier example of Young) that sound was not restricted to them, however lucid the classic mathematical analysis of their motion dating back to Leonhard Euler.[23] Where Young had reduced sound to pure puffs of air, without any vibrating body as their source, Helmholtz used the nascent technology of sirens to "mechanize" this process. He began with such instruments as the Seebeck siren, which used a rotating disc to interrupt an air-stream to produce its wails (figure 9).[24] Though he did not invent this instrument, Helmholtz explored and exploited its implications far beyond earlier investigators,

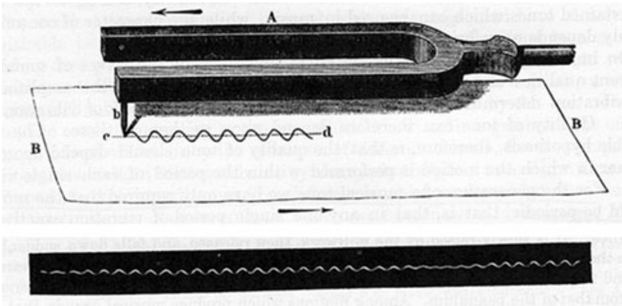

Fig. 8. (*upper*) "To render the law of such motions more comprehensible to the eye than is possible by lengthy verbal descriptions." (*lower*) "[T]his wavy line once drawn, remains as a permanent image of the kind of motion performed by the end of the fork during its musical vibrations." *Source*: Helmholtz, *Die Lehre von den Tonempfindungen*. Zweite Ausgabe (ref. 31), pp. 33-34; *On the Sensations of Tone* (ref. 25), p. 20.

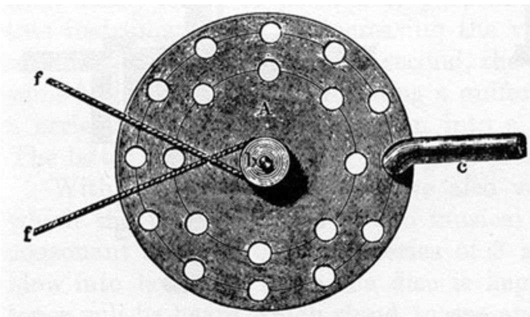

Fig. 9. Seebeck siren, c shows the source of the airstream that is periodically interrupted by the holes in disc A, which the cord f rotates. *Source*: Helmholtz, *Die Lehre von den Tonempfindungen*. Zweite Ausgabe (ref. 31), p. 21; *On the Sensations of Tone* (ref. 25), p. 11.

particularly because he understood the *theoretical* implications of its construction and operation:

> *The sensation of a musical tone is due to a rapid periodic motion of the sonorous body; the sensation of a noise to non-periodic motions….*
>
> ....
>
> [The siren] is constructed in such a manner as to determine the pitch number of the tone produced, by a direct observation....
>
> It is clear that when the pierced disc of one of these sirens is made to revolve with a uniform velocity, and the air escapes through the holes in puffs, the motion of the air thus produced must be *periodic* in the sense already explained. The holes stand at equal intervals of space, and hence on rotation follow each other at equal intervals of time. Through every hole there is poured, as it were,

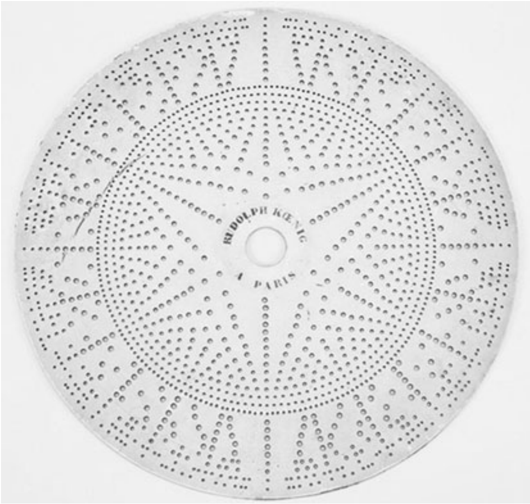

Fig. 10. Disc for an Oppelt siren, made by Rudolph Koenig (1832-1901) *ca.* 1865. *Credit*: Collection of Historical Scientific Instruments, Harvard University.

> a drop of air into the external atmospheric ocean, exciting waves in it, which succeed each other at uniform intervals of time, just as was the case when regularly falling drops impinged upon a surface of water.[25]

Helmholtz, like Young before him, understood that music and noise formed a continuum, distinguished by periodicity of the sound, or lack thereof. The siren renders this periodicity manifest and visual because we see it in the pierced disc whose rotations modulate the airstream: “equal intervals of space” between holes directly generate “drops” of air over “equal intervals of time,” audible as a pure tone. Thus, Helmholtz uses the siren to map visible hole spacings into audible pitches (figure 10), bridging space and time through the spinning disc and the concept of *frequency*, both as the siren’s rotational frequencies and the sound frequencies its disc thereby generates. Helmholtz also advanced the technology of the siren so that it could sound two pitches simultaneously, making possible comparisons in perception (figure 11). Such a double siren could produce “combination tones,” sounding the difference or sum of two pitches, more powerfully than any other instrument. Helmholtz himself discovered the faint sum tones, which he could only produce using a siren or special harmonium; the stronger difference (or “Tartini”) tones had long been known. Helmholtz argued that “the greater part of the force of the combinational tone is generated in the ear itself,” which combines the pure superposition of the incoming pitches, heard as two distinct tones, with their difference or sum, as predicted by nonlinear differential equations derived from Newtonian mechanics.[26] Helmholtz’s use of mathematics shows its essential role in his argument here and in acoustics in general, as he

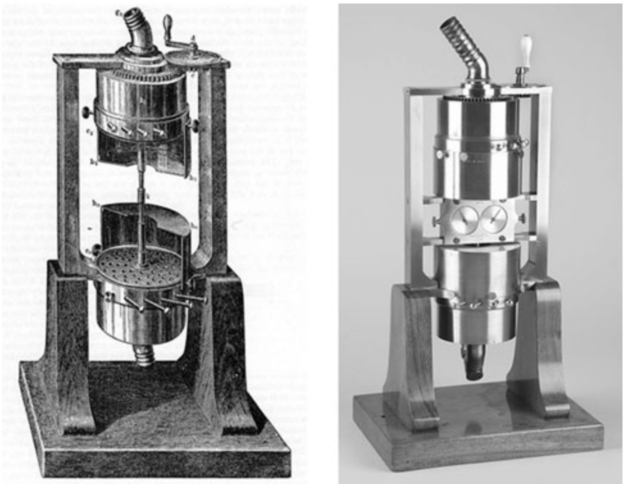

Fig. 11. (*left*) Helmholtz's double siren. *Source*: Helmholtz, *Die Lehre von den Tonempfindungen*. Zweite Ausgabe (ref. 31), p. 242; *On the Sensations of Tone* (ref. 25), p. 162.; (*right*) a double siren built by the Berlin instrument maker Sauerwald *ca.* 1870. *Credit*: Collection of Historical Scientific Instruments, Harvard University.

conceives it. Helmholtz ascribed the failure of superposition and the resultant combination tones to "the unsymmetrical form of the [ear]drum skin itself" and, more importantly, "the loose formation of the joint between the hammer and anvil" ossicles of the middle ear. "In this case, the ossicles may *click*," which he hears as a "mechanical tingling in the ear" when "two clear and powerful soprano voices executed passages in Thirds, in which case the combinational tone comes out very distinctly."[27] Here, his musical experience impinges strongly on the formation of his mathematical acoustics.

Using the double siren, Helmholtz could produce other varieties of "intermittent" or "beat tones," whose sum or difference lies below the frequencies of audible pitches (now called infrasound), hence not hearable as a combination tone but felt viscerally as "a jar or rattle." Such subsonic phenomena probe the differences between hearing and seeing:

> A jarring intermittent tone is for the nerves of hearing what a flickering light is to the nerves of sight, and scratching to the nerves of touch. A much more intense and unpleasant excitement of the organs is thus produced than would be occasioned by a continuous uniform tone.
>
> ....
>
> When the separate luminous irritations follow one another very quickly, the impression produced by each one lasts unweakened in the nerves till the next supervenes, and thus the pauses can no longer be distinguished in sensation. In the eye, the number of separate irritations cannot exceed 24 in a second without

> being completely fused into a single sensation. In this respect the eye is far surpassed by the ear, which can distinguish as many as 132 intermissions in a second and probably even that is not the extreme limit….
>
> The ear is greatly superior in this respect to any other nervous apparatus. It is eminently the organ for small intervals of time, and has been long used as such by astronomers.[28]

This striking comparison shows how far he took comparisons between hearing and seeing to illuminate their shared domains of space and time.

In an 1868 essay on "The Recent Progress of the Theory of Vision," Helmholtz drew attention to another fundamental contrast: vision blends several incoming colors into one perceived hue, but hearing always leaves several notes distinctly *separate*:

> The eye cannot tell the difference, if we substitute orange for red and yellow; but if we hear the notes C and E sounded at the same time, we cannot put D instead of them, without entirely changing the impression upon the ear….
>
> The practiced musician is able to catch the separate notes of the various instruments among the complicated harmonies of an entire orchestra, but the optician cannot directly ascertain the composition of light by means of the eye; he must make use of the prism to decompose the light for him.[29]

Unaided hearing, then, can perceive the precise underlying mathematical ratios within a certain harmony in ways that sight cannot perform without auxiliary instruments. Thus, his 1857 essay, "On the Physiological Causes of Harmony in Music," apostrophized:

> Mathematics and music! the most glaring possible opposites of human thought! and yet connected, mutually sustained! It is as if they would demonstrate the hidden consensus of all the actions of our mind, which in the revelations of genius makes us forefeel unconscious utterances of a mysteriously active intelligence.[30]

Because of the ear's direct access to these mathematical underpinnings, Helmholtz did not rely completely on such mechanical devices as the siren, as useful as they are for isolating and illustrating the periodicities that underlie pitch. He constantly turned back to music itself as his touchstone of sonic experience, to which all his other experiments and speculations refer. As noted above, a sizable part of *Tonempfindungen* is devoted to a rather technical exposition of musical theory, including the sophisticated harmonies of augmented sixth chords that were important in the contemporary music of Wagner and Brahms.* Among the deductions Helmholtz made from music theory, quite apart from acoustics, is a

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* For instance, the famous Prelude to Wagner's *Tristan und Isolde* (first performed in 1865) begins with an augmented sixth chord.

principle that discerns resemblance through grasping what remains *invariant* in different instances:

> We recognise the resemblance between the faces of two near relations, without being at all able to say in what the resemblance consists….
>
> When a father and daughter are strikingly alike in some well-marked feature, as the nose or forehead, we observe it at once, and think no more about it. But if the resemblance is so enigmatically concealed that we cannot detect it, we are fascinated, and cannot help continuing to compare their countenances. And if a painter drew two such heads having, say, a somewhat different expression of character combined with a predominant and striking, though indefinable, resemblance, we should undoubtedly value it as one of the principal beauties of his painting….
>
> Now the case is similar for musical intervals. The resemblance of an Octave to its root is so great and striking that the dullest ear perceives it; the Octave seems to be almost a pure repetition of the root, as it, in fact, merely repeats a part of the compound tone of the root, without adding anything new.[31]

This passage comes in the final pages of the work, in its section on "Aesthetic Relations," as it stood in the first two editions of the book (1863, 1865). We will shortly return to his later (1870) additions that amplify his image; here already Helmholtz recognizes a special quality of spatial "resemblance" or "recurrence" in related shapes and musical intervals that are aesthetically fascinating even (or especially) when "enigmatically concealed." This quest echoes Helmholtz's favorite citation from Friedrich Schiller's poem, "Der Spatziergang": the wise man "seeks a stable pole amid the flight of phenomena" (*sucht den ruhenden Pol in der Erscheinungen Flucht*).[32] Following this advice, Helmholtz sought the stability of invariance in the welter of visual and musical forms.

Riemann's Work on Space and Hearing

Already in 1862, in the midst of his detailed investigations of vision and hearing, Helmholtz became interested in the more general problem of the nature and fundamental character of space itself.[33] At first, he was unaware of the seminal work done decades before by Carl Friedrich Gauss and Bernhard Riemann (figure 12), to which we now turn. Beginning with practical problems in geodesy that originated partly in his work surveying the duchy of Brunswick, in 1827 Gauss had formulated a mathematical criterion that calculated the degree of curvature of a two-dimensional surface (its *intrinsic* or *Gaussian curvature*) using only surveying data collected within that surface.[34] Gauss proved the "remarkable theorem" (*theorema egregium*) that this curvature is *invariant* no matter what coordinate system is chosen in the surface.

In his 1854 *Habilitation* lecture, "On the Hypotheses that Lie at the Foundations of Geometry," Riemann generalized these ideas to what he called a

Fig. 12. Bernhard Riemann (1826-1866) in 1862. *Source*: Riemann, *Gesammelte Mathematische Werke* (ref. 35), frontispiece.

"manifold" (*Mannigfaltigkeit*) having an arbitrary number of dimensions, not just the two dimensions Gauss had considered.[35] Riemann drew the term "manifold" from Kant, who had already used it in his first published work, "Thoughts on the True Estimation of Living Forces" (1747), continuing through his celebrated discussion of space and time in his *Critique of Pure Reason*.[36] Riemann's lecture ends by indicating that his argument leads from geometry and its hypotheses "into the domain of another science, the realm of physics."[37]

Riemann based his argument on a comparison between manifolds, which he defined as comprising "multiply extended quantities," such as the coordinates of ordinary space generalized to arbitrary dimensions or the parameters describing the mixture of colors:

> [T]he general concept of multiply extended quantities, which include spatial quantities, remains completely unexplored….
>
> [O]pportunities for creating concepts whose instances form a continuous manifold occur so seldom in everyday life that color and the position of sensible objects are perhaps the only simple concepts whose instances form a multiply extended manifold.[38]

Though he does not make explicit his sources, we shall see later that Riemann was probably referring to Helmholtz's early 1852 paper on color vision, as well as to

Young's seminal work.* Riemann's wording also raises the question whether or not the manifold of color perception is Euclidean in its geometry, though he does not make this explicit. His general concept of manifold is (as he shortly makes clear) not restricted only to Euclidean geometries but also to the non-Euclidean possibilities that had been revealed decades before by the work of Gauss, Nicolai Ivanovich Lobachevsky, and János Bólyai.

Indeed, the whole point of his lecture is to show that Gauss's concept of intrinsic curvature can be carried forward into manifolds of more than three dimensions. To do so, Riemann generalized the Pythagorean theorem to express the "line element," the invariant length of a line in the multidimensional manifold:

> The problem then is to set up a mathematical expression for the length of a line….
>
> In space, if one expresses the location of a point by rectilinear coordinates, then $ds = \sqrt{\sum (dx)^2}$ [according to the Pythagorean theorem]; [Euclidean] space is therefore included in this simplest case. The next simplest case would perhaps include the manifolds in which the line element can be expressed as the fourth root of a differential expression of the fourth degree. Investigation of this more general class would actually require no essentially different principles, but it would be rather time-consuming and throw proportionally little new light on the study of space, especially since the results cannot be expressed geometrically; I consequently restrict myself to those manifolds where the line element can be expressed by the square root of a differential expression of the second degree.[39]

Thus, as the prime hypothesis on which geometry rests Riemann chose the simplest possible generalization of the quadratic Pythagorean line element ds^2, namely a generalized quadratic form (which may involve more complicated cross-products than the ordinary Pythagorean result).** He mentioned other, less simple possibilities (such as ds^2 being a fourth-degree expression), but set them aside *ex hypothesi* from his subsequent arguments, noting that investigating them would be "time consuming" and implicitly suggesting their relative unimportance for his investigation. Riemann conjectured that any such generalization beyond the quadratic would be "not geometric" and noted that, in a generalized manifold, the

* In 1854, James Clerk Maxwell made the first color photograph, showing the wide range of contemporary work on the consequences of the Young-Helmholtz three-color theory of color perception, referred to in the first edition of Helmholtz, *Handbuch der physiologischen Optik* (ref. 10), pp. 288–297.

** Specifically, Riemann generalizes the quadratic Euclidean line element $ds^2 = dx_1^2 + dx_2^2 + dx_3^2$ (in terms of three spatial coordinates called, for convenience, x_1, x_2, x_3) to a general quadratic form $ds^2 = g_{11}\, dx_1^2 + g_{12}\, dx_1 dx_2 + g_{22}\, dx_2^2 + \cdots = \Sigma g_{\mu\nu}\, dx^\mu dx^\nu$ (summed over all n dimensions, $\mu, \nu = 1, \ldots, n$). In this modern notation (due to Einstein, following Levi-Civita), $g_{\mu\nu}$ is the "metric tensor." Note also that Riemann did not include these calculations in his 1854 lecture, though they did appear in his paper.

geometry in an infinitesimal neighborhood around any point is Euclidean, so that *locally* the geometry of the general manifold must reduce to Euclidean "flat" space. Riemann took this local limit as a definitive sign of the global quadratic form of the manifold's metric (to use modern terminology), meaning its distance relations.

The implications of this visionary lecture excited and startled its 1854 audience, including Gauss himself, who had chosen this very topic from Riemann's list of proposals. Between then and his death from tuberculosis at the age of forty, Riemann worked intensively on several projects. He had made important strides in understanding electromagnetism and in 1858 was the first to formulate a partial differential equation expressing the propagation of the electric potential with the velocity of light, thus providing an electrodynamic wave equation.[40] By comparison, James Clerk Maxwell only derived such an equation in 1868, *after* having set forth the field equations that today bear his name and having duly acknowledged Riemann's priority.[41] Yet Riemann was able to reach his wave equation without having completed what, for Maxwell, was necessary groundwork.

It is tempting to speculate that Riemann might have been able to complete an independent deduction of the full electromagnetic field theory, had he lived longer. As it was, his wave equation explicitly linked the *time* and *space* dependence of the electric potential. His 1854 lecture had positioned him to consider higher-dimensional manifolds; his electromagnetic wave equation offered him a link between the "dimensions" of space and time. Still, during the period 1854–1861 he produced the mathematical work on distribution of prime numbers and the zeta function, now called the Riemann Hypothesis (1859), probably his most famous initiative and the premier unsolved mathematical problem up to the present day.[42] This, by itself, might help explain why he might not have placed electromagnetism higher in his list of priorities, though his surviving drafts and papers show his continuing interest in physics, not to speak of his other important mathematical projects. The speculative writings that remain as fragments in his posthumous papers show that his attention in natural philosophy was directed to the possible unification of gravitation and electricity.[43] Given the general framework of his 1854 lecture, Riemann's project seems to have envisaged using his many-dimensional curved manifolds as the framework for a unified theory of all physical forces.

These theoretical drafts give the context for what remained, at his death, his major uncompleted paper on "The Mechanism of the Ear."[44] For both Riemann and Helmholtz, the problem of hearing was a significant part of their larger enterprises, an intermediate zone in which waves, geometry, and sensation met. Riemann's choice to study the ear (rather than the eye) is also noteworthy; surely questions of hearing must have seemed very important to him if he set them next to or even ahead of his other ambitious projects in electrodynamics, gravitation, and number theory. By comparison with Helmholtz, little evidence survives that would give biographical insight into Riemann's choice. The son of a pastor and himself deeply religious, Riemann considered "daily self-examination before the face of God" to be "the main point in religion." Alongside this austere, contemplative persona, Riemann evidenced considerable love of art. According to his friend Richard Dedekind, Riemann's long stays in Italy after 1862, seeking to recover his health, "were a true luminous point in his life … looking at the glory of this enchanting land, of nature and art, made him endlessly happy." The newly married Riemann took "great interest" in the "art treasures and antiquities" of Italy, also greatly admired by other *Kulturträger*, such as Helmholtz.[45] Like most of them, Riemann probably felt deeply the power of music.

At any rate, Riemann's deep interest in understanding the ear shines through his essay. Riemann praises Helmholtz's ingenious experimental work on hearing, while criticizing his findings and basic methodology. In Riemann's view, Helmholtz *synthesizes* the anatomical structures of the ear into the functioning of the whole organ, but at the cost of making questionable teleological assumptions about those structures. Instead, Riemann advocates an alternative process of *analysis* that begins with the observed behavior of the whole organ and then constructs a mathematical model that would explain those functions as necessary, not merely sufficient. By emphasizing the central functions of the organ *in toto*, Riemann strives to avoid Helmholtz's suppositions about the teleological interrelation of its anatomical subunits. Riemann uses anatomical knowledge for clues to guide his model building, not as a definitive level of explanation.

The post-Kantian language of analysis and synthesis, the contrast between necessity and sufficiency, mark Riemann's approach as essentially mathematical and hypothetical in spirit.

> We do not—as Newton proposes—completely reject the use of analogy (the "poetry of hypothesis"), but rather afterwards emphasize the conditions that *must* be met to account for what the organ accomplishes, and discard any notions that are not essential to the explanation, but that have arisen solely through the use of analogy.[46]

In contrast to Newton's famous strictures against "feigning" hypotheses, Riemann's remarkable expression, "the 'poetry of hypothesis' [*Dichten von Hypothesen*]," rhetorically emphasizes the creative freedom of imagination, its suggestive power in the formation of analytic representations of phenomena, whether aural or geometric, in the form of hypotheses that are not restricted by anatomical suppositions.

With this in mind, we can read Riemann's "Mechanism of the Ear" as a nascent essay "On the Hypotheses that Lie at the Foundations of Hearing," comparable to his earlier work on the hypotheses he considered fundamental to geometry. Enough remains of Riemann's draft to show some general features of his proposed analysis. Against Helmholtz's assertion that the ossicles click, Riemann notes that:

> The apparatus within the tympanic cavity (in its unspoiled condition) is a mechanical apparatus whose sensitivity is infinitely superior to everything we know about the sensitivity of mechanical apparatuses.
>
> In fact, it is by no means improbable that it faithfully transmits sonic motions that are so small that they cannot be observed with a microscope.[47]

For instance, "the call of the Portsmouth sentry is clearly audible at night at a distance of 4 to 5 English miles," so that "the ear does pick up sounds whose mechanical force is millions of times weaker than that of sounds of ordinary intensity." This, he feels, negates Helmholtz's claim about the noisiness of the ossicles, which Riemann judges a teleological supposition introduced primarily to support Helmholtz's theory of combination tones.

Instead, Riemann's approach is much closer to what now is called systems theory: he treats the ear as a "black box" whose overall functioning can be mathematically modeled based on its essential phenomenological parameters, especially its high sensitivity and fidelity.[48] His modeling involves pointed comparisons with vision:

> I find nothing whatsoever [in hearing] analogous to the eye's response to the degree of illumination of the visual field, and have no idea what a continuously variable reflex activity of *M. tensor tympani* is supposed to contribute to the exact comprehension of a piece of music.[49]

Here Riemann refers to the tensor tympani muscle that attaches to the hammer bone of the middle ear and can dampen the vibrations of the tympanic membrane. Though Helmholtz had not explicitly extended his sign theory to hearing, Riemann seems to take him to imply that the varying states of the tympanic muscle are "local signs" of the associated sounds, as the movements of the eye muscles are signs of what it sees.[50] If so, the variable activity of the tensor tympani muscle would correlate with auditory response to varying musical sounds. In contrast, Riemann argues that a *constant* tension of this auditory muscle should accompany the activity of "the alert ear—the ear deliberately prepared for precise perception," whose acuity depends on the tympanic muscle to maintain steady contact between the ossicles and the inner ear.[51]

Riemann's analytic program requires that "we must now derive from the empirically known functions performed by the organ, the conditions which must be met in this transmission … [by] seeking a mathematical expression for the nature of the pressure fluctuation upon which timbre depends."[52] Though in his 1854 lecture Riemann held that "color and the position of sensible objects are perhaps the only simple concepts whose instances form a multiply extended manifold," by 1866 he seems poised to treat hearing as a further example of such a manifold. Riemann does not provide any mathematical details of his approach to hearing, but, based on his work on geometrical "hypotheses" and his work on shock waves in fluids, we may infer that he intended to use some kind of multidimensional manifold, analogous to those he proposed to represent the geometric effect of physical sources.[53] Where Helmholtz took evidence from hearing and seeing into his geometric investigations, Riemann traversed an opposite course, applying geometric insights to model the functioning of the ear.

In its unfinished form, Riemann's "Mechanism of the Ear" was published posthumously in a medical journal in 1867. Helmholtz responded in 1867 and 1869 in two papers, "On the Mechanism of the Ossicles of the Ear," whose titles once again reflect the fundamental contrast between the two men: Helmholtz's "facts" (or ossicles) *versus* Riemann's "poetry of hypothesis" (the ear considered as a high-sensitivity sound transducer, regardless of its anatomical details).[54] Though publicly Helmholtz wrote respectfully of the "great mathematician's" foray into his own domain, privately he expressed irritation at Riemann the "amateur."[55] In his printed response, Helmholtz did not engage Riemann's philosophical contrast between analytic and synthetic, but argued that the ossicles can act "practically, as absolutely solid bodies" that thereby can transmit sound with the high sensitivity Riemann had emphasized. To show that his anatomical model could meet Riemann's critique, Helmholtz gave a detailed account of the fine structure of the ossicles and their subtle interconnections, as well as of the tensor tympani muscle (figure 13). Rhetorically, Helmholtz sweeps away Riemann's theorizing under a deluge of anatomic observations, implicitly arguing that only in such terms can any physiology of the ear be responsibly phrased. For the time being, Riemann, the defunct "amateur," was quietly buried under a mountain of Helmholtz's "professional" anatomy.*

Helmholtz and the "Problem of Space"

This controversy about hearing led Helmholtz to devote much attention to Riemann's work, yet he only received Riemann's 1854 lecture in May 1868, the year in which it finally appeared in print.[56] Yet even *before* he had read it, Helmholtz had already inferred "that Riemann came to exactly the same conclusion as myself," as he wrote Ernst Schering on April 21, 1868.

> My starting-point is the question: What must be the nature of a magnitude of several dimensions in order that solid bodies (i.e. bodies with unaltered relative

* Helmholtz's detailed description of the ear was superseded by later anatomical findings, particularly because the larger context of the processing of hearing became understood as involving the auditory system of the brain as well. Rather than being a kind of nerve-piano, its separate cilia sympathetically responding to incoming pitches, the cochlea currently is considered to comprise a series of chambers of variable resonant frequency, in which the cilia respond to the local amplitude of vibration, rather than its frequency. As Riemann surmised, the overall functioning of hearing may be described in terms of inputs and outputs of a complex electrical network. See, for example, Jonathan Sterne, *The Audible Past: Cultural Origins of Sound Reproduction* (Durham and London: Duke University Press, 2003), pp. 62–67.

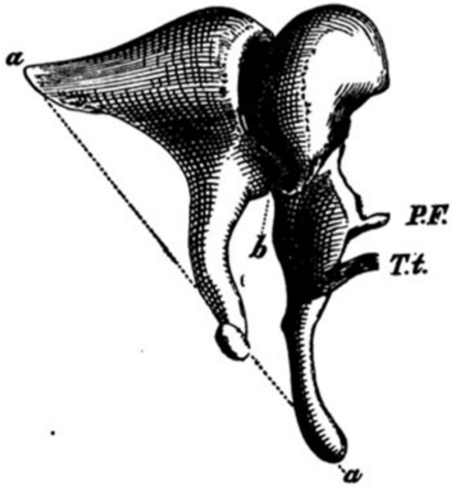

Fig. 13. Sketch showing Helmholtz's response to Riemann regarding the precise functioning of the hammer and anvil. *Source*: Helmholtz, "Mechanik der Gehörknöchelchen und des Trommelfelles" (ref. 54), p. 557; *Mechanism of the Ossicles of the Ear* (ref. 54), p. 45.

> measurements) shall everywhere be able to move in it as continuously, monodromously, and freely, as do bodies in actual space? Answer, expressed according to our analytical geometry: "Let x, y, z, t be the rectangular co-ordinates of a space of four dimensions, then for every point of our tridimensional space it follows that $x^2 + y^2 + z^2 + t^2 = R^2$, where R is an undetermined constant, which is infinite in Euclidean space."[57]

This extraordinary statement has received little notice, though (to my knowledge) it may be the first explicit use of four dimensions to state and address what became called "the problem of space," aside from a few speculative remarks by Jean le Rond d'Alembert (1754) and Joseph Louis Lagrange (1797).[58] As we shall see, the formulation given in this letter remained in Helmholtz's mind.

Helmholtz's pursuit of invariance, whether as resemblances in the visual field or as recurrences in music, led directly to his 1868 paper, "On the Factual Foundations of Geometry," which begins with an explicit connection to his work on the physiology of vision:

> Investigations into how localization in the visual field comes to pass have led the author also to reflect on the origins of spatial intuition in general. This leads first of all to a question whose answer definitely belongs to the sphere of exact science, namely, which propositions of geometry express truths of factual significance and which, on the contrary, are only definitions or consequences of definitions and their particular manner of expression?…

> [O]ne could follow this direction and find out which analytical characteristics of space and spatial magnitudes must be presupposed in order to ground the propositions of analytic geometry completely from the beginning.[59]

For Helmholtz, questions about "the origins of spatial intuition in general" emerge from studies of the visual system and lead directly to considerations about the nature of geometry. In so doing, he breached the customary barrier between the propositions of geometry and physical reality, previously considered separate from one another.

Though during this period he had been mainly working on experimental physiology, Helmholtz reveals that he had gone remarkably far in his own self-directed reconsideration of the mathematical and philosophical problems concerning the nature of space:

> The author had already begun such an investigation and had completed it in the main when Riemann's habilitation lecture "On the Hypotheses That Lie at the Foundations of Geometry" was made public, in which an identical investigation is carried out, having only a slightly different formulation of the question. On this occasion, we learned that Gauss had also worked on the same subject matter, of which his famous essay on the curvature of surfaces is the only published part of that investigation.[60]

Riemann's argument assumes a generalized quadratic line element but does not prove its necessity. Helmholtz asked whether there is some fundamental reason that would necessarily mandate this assumption, rather than other, more general possibilities.

Helmholtz's 1868 paper summarized his response to this problem.[61] Though he shared with Riemann the fundamental idea that geometry ultimately rested on physics, rather than on transcendental ideas, Helmholtz replaced Riemann's "hypotheses" with "facts." Steeped in Goethe, like his educated contemporaries, Helmholtz knew by heart Faust's amendment of the Gospel of St. John's opening line from "In the beginning was the Word" to "In the beginning was the Deed" ("*Im Anfang war die That*"); like Faust, Helmholtz moved from the Word (or Riemann's "hypotheses") to the Deed, understood as the Fact.[62]

Helmholtz argued that fundamental physical facts necessitate the quadratic form of the line element. Specifically, he assumes: "(1) *Continuity and dimensions*" (each point in space is determined by *n* continuous, independent variables); "(2) *The existence of moving and rigid bodies*"; "(3) *Free mobility*" (meaning that "each point can pass over into any other along a continuous path"); and finally "(4) *The invariance of the form of rigid bodies under rotation*."[63] From these premises, he deduced that

> if we desire to find the degree of rigidity and mobility of natural bodies attributable to our space in a space of otherwise unknown properties, the

> square of the line element *ds* would have to be a homogenous second-degree function of infinitely small increments of the arbitrarily chosen coordinates *u, v, w*. This proposition … [is] the most general form of the Pythagorean Theorem.* The proof of this proposition vindicates the assumption of Riemann's investigations into space.[64]

In his original draft, Helmholtz thought this meant that the quadratic form had to correspond to *Euclidean* geometry, but Eugenio Beltrami and Sophus Lie soon objected to this erroneous overspecialization of a more generalized result that Helmholtz should have found: in fact, the quadratic form was, in general, non-Euclidean, as Helmholtz acknowledged in a note appended to his 1868 paper.[65] In the subsequent literature, this issue became known as the Helmholtz-Lie *Raumproblem*, the "problem of space" *par excellence*; not a merely technical matter or a fine point of mathematical rigor, this problem has deep implications for Einstein's geometric account of gravitation because it dictates the fundamental form of the metric, the geometrical field created by the bodies immersed in it, which in turn move along its shortest (geodesic) paths.[66]

Helmholtz's oversight probably implies his initial lack of knowledge about non-Euclidean geometry, confirming that he was not aware of the non-Euclidean import of his color diagrams (figure 5) when he first published them in 1867, before his enlightenment by Beltrami and Lie.[67] In his 1868 paper, he emphasized that:

> The independence of the congruence of rigid point-systems from place, location, and the system's relative rotation is the fact on which geometry is grounded.
>
> This becomes even clearer when we compare space with other multiply extended manifolds, for example the system of colors. In this case, as long as we have no other method of measurement than through the law of color mixing, there exists, unlike in space, no relation of magnitudes between any two points that can be compared with that between two other points. Instead, there exists a relation between groups of any three points that also must lie in a straight line (that is, in groups of any three colors, among which any one is mixable into the other two).
>
> We find another difference in the field of vision of a single eye, where no rotations are possible so long as we confine ourselves to natural eye movements.[68]

Under the influence of Riemann's conception of manifold, Helmholtz now reinterprets his earlier diagrams of "the system of colors" as a "threefold-extended manifold" comparable to three-dimensional space (*Raum*).[69] Though

* This is the form for *ds* given in footnote **, under the section "Riemann's Work on Space and Hearing."

we have become used to the notion that non-spatial magnitudes can be described as if they constituted a "space," this broadening of the concept of space should be credited to Riemann's manifolds.* Following on Helmholtz's pioneering experimental studies of vision, Riemann adduced space and color as comparable manifolds, terminology Helmholtz then used to categorize the "system of colors" more deeply.

Helmholtz's Musical "Space"

Overlapping his work on this "problem of space," Helmholtz returned to musical concerns as he prepared a third edition of his *Tonempfindungen* (1870). His new additions clarify the significance of music for his thinking about geometry as he developed his nascent ideas of resemblance and invariance. In the 1870 version, he expanded the concluding passage of the work concerning visual resemblances and musical recurrences, adding that they should be regarded as "by no means a merely external indifferent regularity"; in contrast, poetic rhythm is an "external arrangement" merely imposed on words to make them conform to metrical units. Instead,

> [I have shown] that the equality of two intervals lying in different sections of the [musical] scale would be recognized by immediate sensation…. This produces a definiteness and certainty in the measurement of intervals for our sensations, such as might be looked for in vain in the system of colours, otherwise so similar, or in the estimation of mere differences of intensity in our various sensual perceptions.[70]

The invariance of musical intervals or melodies, when transposed, has no precedent in the "space" of color; we can transpose a Beethoven sonata up a half step and still recognize the work as in some sense still the same, yet we cannot likewise "transpose" all the colors of a Rembrandt (say by shifting all reds to orange, orange to yellow, and so forth): Though the basic line and surface contours of the painting remain unchanged, its color harmony cannot be "transposed" and remain recognizably identical. Helmholtz extends the special quality of spatial resemblance that can be seen in related shapes (such as the similar profiles of father and daughter) to the characteristic melodic contour of a certain piece of music, but *not* to colors.

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* The influence of Grassmann should also be considered, though that by itself does not seem to have been sufficient for Helmholtz to speak of manifolds in his 1867 *Handbuch*, which does mention Grassmann. Riemann does not seem to have known Grassmann's work, which still is "not a general theory of manifolds," as argued by Torretti, *Philosophy of Geometry* (ref. 66), p. 109. See also Erhard Scholz, *Geschichte des Mannigfaltigkeitsbegriffs von Riemann bis Poincaré* (Boston, Basel, Stuttgart: Birkhäuser, 1980), esp. pp. 24–94, 113–123.

Helmholtz goes on to emphasize the consequences of this resemblance or invariance in music:

> Upon this reposes also the characteristic resemblance between the relations of the musical scale and of space, a resemblance which appears to me of vital importance for the peculiar effects of music. It is an essential character of space that at every position within it like bodies can be placed, and like motions can occur. Everything that is possible to happen in one part of space is equally possible in every other part of space and is perceived by us in precisely the same way. This is the case also with the musical scale. Every melodic phrase, every chord, which can be executed at any pitch, can be also executed at any other pitch in such a way that we immediately perceive the characteristic marks of their similarity. On the other hand, also, different voices executing the same or different melodic phrases, can move at the same time within the compass of the scale, like two bodies in space, and, provided they are consonant in the accented parts of bars, without creating any musical disturbances. Such a close analogy consequently exists in all essential relations between the musical scale and space, that even alteration of pitch has a readily recognized and unmistakable resemblance to motion in space, and is often metaphorically termed the ascending or descending *motion* or *progression* of a part. Hence, again, it becomes possible for motion in music to imitate the peculiar characteristics of motive forces in space, that is, to form an image of the various impulses and forces which lie at the root of motion. And on this, as I believe, essentially depends the power of music to picture emotion.[71]

Because music relies on the recognition of analogy, resemblance, and invariance, Helmholtz deduces that it therefore can "imitate the peculiar characteristic of motive forces in space": Though not itself spatial or extended, music can *move* in precise analogy to spatial motion, from which Helmholtz boldly identifies the emotive force of music: its virtual motion is felt as emotion precisely because of the deep isomorphism between musical and physical space.

Musical, Visual, and Geometric Manifolds

Over the next few years, Helmholtz extended the implications of his 1868 arguments about the *Raumproblem*. He presented popular lectures and essays, addressed to a wider educated audience, concerning larger philosophical issues emergent from his own work.[72] Immediately after completing the additions we have just considered to the 1870 edition of his *Tonempfindungen*, Helmholtz delivered his lecture "On the Origin and Significance of Geometrical Axioms," which discusses "the philosophical bearing of recent inquiries concerning geometrical axioms and the possibility of working out analytically other systems of geometry with other axioms than Euclid's."[73]

For the first time in his writings on the *Raumproblem*, Helmholtz describes in detail non-Euclidean geometries and the pseudosphere of Beltrami, whose criticisms had first moved Helmholtz to address this issue directly.* Helmholtz also brings forward a striking device for comparing and contrasting these different geometries: "Think of the image of the world in a convex mirror." In this mirror-world, the theorems of Euclidean geometry would instantly be translated into non-Euclidean image-theorems, at least as seen from our side of the mirror:

> In short I do not see how men in the mirror are to discover that their bodies are not rigid solids and their experiences good examples of the correctness of Euclid's axioms. But if they could look out upon our world as we can look into theirs, without overstepping the boundary, they must declare it to be a picture in a spherical mirror, and would speak of us just as we speak of them.... [N]either, so far as I can see, would be able to convince the other that he had the true, the other the distorted relations.[74]

As further evidence, Helmholtz also adduces the eye's ability to accommodate when looking through "convex spectacles" with which he had experimented in the course of his visual studies: "[A]fter going about a little the illusion would vanish.... We have every reason to suppose that what happens in a few hours to anyone beginning to wear spectacles would soon enough be experienced in pseudospherical space. In short, pseudospherical space would not seem to us very strange, comparatively speaking,"[75] once we had gotten used to it, just as our eyes would quickly get used to those "distorting" spectacles. Helmholtz's penetrating insight into the relative consistency of these seemingly antithetical geometries, Euclidean and non-Euclidean, is directly indebted to his studies of visual physiology. Indeed, looking back at his recent work, he remarks that:

> Whilst Riemann entered upon this new field from the side of the most general and fundamental questions of analytical geometry, I myself arrived at similar conclusions, partly from seeking to represent in space the system of colours, involving the comparison of one threefold extended manifold with another, and partly from inquiries on the origin of our ocular measure for distances in the field of vision.[76]

As in his 1868 paper, Helmholtz considers that his own "facts" confirm Riemann's "hypotheses."

In his 1870 exposition, besides adducing the three-dimensional manifolds of "the space in which we live" and "the system of colors," Helmholtz adds that: "[T]ime also is an aggregate [a manifold] of one dimension."[77] Here, for the first time, *time* enters the discussion as another possible manifold, albeit one-dimensional.[78] Nor did Riemann include time explicitly in his geometrical (hence

* A pseudosphere is a surface of constant negative curvature, roughly saddle-shaped, used by Beltrami in 1868 as a model for Lobachevky's hyperbolic geometry.

implicitly spatial) manifolds. Immediately after his mention of time, Helmholtz goes on to include the manifold of musical tones, whose time dependence he had studied so closely.

> In the same way we may consider the system of simple tones as an aggregate [a manifold] of two dimensions, if we distinguish only pitch and intensity and leave out of account differences of timbre. This generalization of the idea is well-suited to bring out the distinction between space of three dimensions and other aggregates [manifolds]. We can, as we know from daily experience, compare the vertical distance of two points with the horizontal distance of two others, because we can apply a measure first to the one pair and then to the other. But we cannot compare the difference between two tones of equal pitch and different intensity with that between two tones of equal intensity and different pitch. Riemann showed by considerations of this kind that the essential foundation of any system of geometry is the expression that it gives for the distance between two points lying in any direction from one another….[79]

Helmholtz's concept of manifold includes music and sound in the same arena as space, time, color, and vision. Though *simple* tones may be described as a manifold of two dimensions, Helmholtz had investigated the parameters of timbre that distinguish complex musical sonorities from simple tones. At this point, the question of dimensionality seems open: going beyond the two dimensions of simple tones, how many dimensions really are needed to describe the full character of musical "space"? And what then of the dimensional relations between space and time?[80] Though he does not go further, Helmholtz leaves the *Raumproblem* as the shared heritage of the manifolds of music, vision, space, and time.

The conclusion of Helmholtz's 1876 revised version of his essay, "On the Origin and Meaning of Geometrical Axioms," clarifies his current understanding that:

> (1) The axioms of geometry, taken by themselves out of all connection with mechanical propositions, represent no relations of real things…. [T]hey constitute a form into which any empirical content whatever will fit, and which therefore does not in any way limit or determine beforehand the nature of the content. This is true, however, not only of Euclid's axioms, but also of the axioms of spherical and pseudospherical geometry.
>
> (2) As soon as certain principles of mechanics are conjoined with the axioms of geometry, we obtain a system of propositions which has real import, and which can be verified or overturned by empirical observations….[81]

Where Kant had judged Euclidean geometry to be valid *a priori*, Helmholtz included Euclidean and non-Euclidean geometries on the same footing, each "a form into which any empirical content whatever will fit."[82] Hence, the axioms of geometry must meet "certain principles of mechanics" in ways that finally rest on empirical observations. Helmholtz's view of this empirical confrontation was informed both by optics (and visual physiology) and mechanics (and its connection to acoustics and music).

In this revised, 1876 version, Helmholtz also added a mathematical appendix on "the elements of the geometry of spherical space," the same four-dimensional manifold he had mentioned to Schering in his 1868 letter, cited above, described by the expression $x^2 + y^2 + z^2 + t^2 = R^2$. Though he seems to treat t as a fourth spatial coordinate, its common identification as time pervaded contemporary mathematical physics; Helmholtz also allowed t to become an imaginary quantity, further increasing the similarity with the pseudo-Euclidean space-time used by Einstein and Minkowski.*

Such beguiling speculations aside, it would go much too far to conclude that Helmholtz had (even unknowingly) written down an expression from relativistic physics, fifty years in advance. His appendix, however, does illustrate his ability to invoke a four-dimensional manifold to describe mathematically our visual experience, were we looking "through a pair of convex spectacles" specially ground to give a negative focal length. Consonant with his empirical method, Helmholtz showed that we could thereby imagine a four-dimensional pseudospherical "space," contra Kant's denial of that possibility. Long before Edwin Abbott's *Flatland* (1892), in this essay Helmholtz was probably the first writer to describe "reasoning beings of only two dimensions" who "live and move on the surface of some solid body" in order to help us imagine the felt reality of higher dimensions.[83]

Relativistic Resonances

The influence of Helmholtz and Riemann remained crucial in subsequent developments of the problem of space. As noted above, Lie embedded his correction of Helmholtz's erroneous generalizations in the emergent structure of his theory of continuous groups.[84] Aside from William Kingdon Clifford's response,[85] Riemann's work lay dormant among his immediate successors. The philosophical implications of Helmholtz's work were important to Felix Klein in connection with his Erlangen Program to characterize spaces by their characteristic groups of transformations and respective invariants.[86] As Klein remarked in 1893, "our ideas of space come to us through the senses of vision and motion, the 'optical properties' of space forming one source, while the 'mechanical properties' form another; the former corresponds in a general way to the projective properties, the latter to those discussed by Helmholtz."[87]

* Helmholtz writes his four-dimensional line element as $ds^2 = dx^2 + dy^2 + dz^2 + dt^2$, in which he then allows t to become imaginary ($t = i\tau$), so that the four-dimensional manifold is now pseudospherical and hence $ds^2 = dx^2 + dy^2 + dz^2 - d\tau^2$, exactly the form of the Lorentzian line-element used by Einstein and Minkowski, if $\tau = ct$, where c is the speed of light.

Henri Poincaré emphasized Riemann's work and also responded strongly to Helmholtz's arguments in connection with his own view that convention and convenience underlie the choice of a geometry for space.[88] Poincaré also carried forward Helmholtz's thought experiment of viewing the Euclidean world through convex mirrors or distorting spectacles, which Poincaré phrased in terms of a "dictionary" that would translate the terms of Euclidean geometry into non-Euclidean terms, one for one, so as to make clear that Euclidean geometry was no less consistent than Lobachevskian.[89] Thus, within purely Euclidean geometry, a model could be made of Euclidean figures that in every respect behaved like Lobachevskian geometry, once the fundamental elements (lines, angles, etc.) had been suitably redefined, corresponding to the action of the distorting mirrors or lenses; conversely, Lobachevskian geometry could be made to behave as if it were Euclidean by a similar set of redefinitions. Poincaré's argument and Klein's further activities in providing other such models were crucial steps in understanding the relationships between the different geometries as not only equally *possible* but equally *consistent*. This demonstrated equality of status in turn opened the possibility of addressing the empirical observations that (as Helmholtz suggested) might then ground the choice between geometries.

Einstein's general theory of relativity gave a precise form to this connection between the empirical (understood as stress-energy) and the geometrical (the invariant curvature of space-time).[90] Rather than ignoring the history of these concepts (as he sometimes is represented), in fact Einstein (figure 14) was deeply

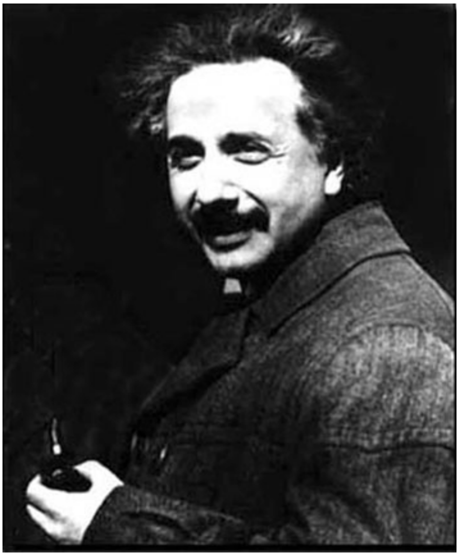

Fig. 14. Albert Einstein (1879-1955). *Credit*: American Institute of Physics Emilio Segrè Visual Archives.

conscious of history and drew not only general inspiration but specific guidance from what went before. As Einstein wrote Robert Thornton in 1944:

> A knowledge of the historic and philosophical background gives that kind of independence from prejudices of his generation from which most scientists are suffering. This independence created by philosophical insight is—in my opinion—the mark of distinction between a mere artisan or specialist and a real seeker after truth.[91]

Einstein's own essays contain a wealth of historical reflection and awareness, such as his observation that:

> Only the genius of Riemann, solitary and uncomprehended, by the middle of the last century already broke through to a new conception of space, in which space was deprived of its rigidity and in which its power to take part in physical events was recognized as possible.[92]

Indeed, Riemann had worked out the curvature tensor (now named after him) that was all important for Einstein's general theory.[93] Einstein's tribute pays what he recognizes as a major debt. Arguably, too, Einstein's famous description of physical theory as "free creations of the human mind" may have its roots in Riemann's "poetry of hypotheses."[94]

Einstein's words in praise of Riemann are far better known than his 1917 encomium of Helmholtz's Goethe essays—"Dear reader! Summarizing would be profanation. Read for yourself!"[95]—or his 1925 *hommage*: "[that] all propositions of geometry gain the character of assertions about real bodies … was especially clearly advocated by Helmholtz, and we can add that without him the formulation of relativity theory would have been practically impossible."[96] Einstein considered Helmholtz's connection of geometric hypotheses to empirical facts as absolutely crucial for the general theory of relativity, whose field equations epitomize that connection. To reach that point, Helmholtz connected his work in music and vision, hearing and seeing, whose comparison lay at the grounds of his synthetic understanding. His dialogue with Riemann reflects and underscores the significance of their shared concern with hearing in the context of the problem of space and the physical foundations of geometry.

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[58] For instance, Helmholtz's statement is not discussed in Alfred M. Bork, "The Fourth Dimension in Nineteenth-Century Physics," *Isis* **55** (1964), 326–338. For the references to d'Alembert and Lagrange, see R.C. Archibald, "Time as a Fourth Dimension," *Bulletin* (*New Series*) *of the American Mathematical Society* **20** (1914), 409–412.

[59] Hermann Helmholtz, "Ueber die thatsächlichen Grundlagen der Geometrie," [1866] in *Wissenschaftliche Abhandlungen*. Zweiter Band (ref. 4), pp. 610–617, on pp. 610–611; translated as "On the Factual Foundations of Geometry," in Pesic, *Beyond Geometry* (ref. 35), pp. 47–52, on p. 47. The date of 1866 given in Helmholtz's *Wissenschaftliche Abhandlungen* seems to have been a misprint for 1868, as shown by Klaus Volkert, "On Helmholtz' Paper 'Ueber die thatsächlichen Grundlagen der Geometrie'," *Historia Mathematica* **20** (1993), 307–309.

[60] Helmholtz, "Ueber die thatsächlichen Grundlagen" (ref. 59), p. 611; "On the Factual Foundations" (ref. 59), pp. 47–48.

[61] In 1868 Helmholtz published another, longer paper that presents the details of the argument he had summarized in his brief 1868 paper and incorporates the corrections that he had learned from Beltrami and Lie; see Hermann von Helmholtz, "Ueber die Thatsachen, die der Geometrie zum Grunde liegen" [1868] in *Wissenschaftliche Abhandlungen*. Zweiter Band (ref. 4), pp. 618–639; reprinted in Hertz and Schlick, *Helmholtz: Schriften zur Erkenntnistheorie* (ref. 50), pp. 38–69; newly translated by Malcolm F. Lowe as "On the Facts Underlying Geometry," in Cohen and Elkana, *Helmholtz: Epistemological Writings,* (ref. 50), pp. 39–71. For a helpful modern account of Helmholtz's argument, see Ronald Adler, Maurice Bazin, and Menahem Schiffer, *Introduction to General Relativity*, Second Edition (New York: McGraw-Hill Book Company, 1975), pp. 7–16; see also B.A. Rosenfeld, *A History of Non-Euclidean Geometry: Evolution of the Concept of a Geometric Space*. Translated by Abe Shenitzer (New York, Berlin, Heidelberg: Springer-Verlag, 1988), pp. 333–338; Renate Wahsner, "Apriorische Funktion und aposteriorische Herkunft: Hermann von Helmholtz' Untersuchungen zum Erfahrungsstatus der Geometrie," in Krüger, *Universalgenie Helmholtz* (ref. 17), pp. 245–259; Olivier Darrigol, "Number and measure: Hermann von Helmholtz at the crossroads of mathematics, physics, and psychology," *Studies in History and Philosophy of Science* **34** (2003), 515–573; *idem*, "A Helmholtzian approach to space and time," *ibid.* **38** (2007), 528–542; Gregor Schiemann, *Hermann von Helmholtz's Mechanism: The Loss of Certainty* [*Archimedes*, Vol. 17] ([Dordrecht]: Springer, 2009), pp. 98–110.

[62] Goethe, *Faust*, Part I, line 1237. Beside many other references, Helmholtz devoted two major essays to Goethe, "Ueber Goethe's naturwissenschaftliche Arbeiten" [1853], in *Vorträge und Reden*. Fünfte Auflage. Erster Band (ref. 7), pp. 23–45; translated by H.W. Eve as "On Goethe's Scientific Researches" in *Popular Scientific Lectures* (ref. 7), pp. 1–21, and "Goethe's Vorahnungen kommender naturwissenschaftlicher Ideen" [1892], in *Vorträge und Reden*. Fünfte Auflage. Zweiter Band (ref, 19), pp. 335–361; former also translated, as well as the latter as "Goethe's Presentiments of Major Scientific Ideas," in David Cahan, ed., *Science and Culture: Popular and Philosophical Essays* (Chicago and London: The University of Chicago Press, 1995), pp. 1–17, 393–412.

[63] Helmholtz, "Ueber die thatsächlichen Grundlagen" (ref. 59), pp. 614–615; "On the Factual Foundations" (ref. 59), pp. 49–50.

[64] *Ibid.*, p. 617; 51.

[65] *Ibid.*, p. 638–639; 51. For the work of Beltrami, see Jeremy Gray, *Ideas of Space: Euclidean, Non-Euclidean, and Relativistic*, Second Edition (Oxford: Clarendon Press, 1989), pp. 147–154.

[66] Roberto Torretti, *Philosophy of Geometry from Riemann to Poincaré* (Dordrecht-Holland and London: D. Reidel Publishing Co., 1978), pp. 155–179; Volkmar Schüller, "Das Helmholtz-Liesche Raumproblem und seine ersten Lösungen," in Krüger, *Universalgenie Helmholtz* (ref. 17), pp. 260–275; Klaus Volkert, "Hermann von Helmholtz und die Grundlagen der Geometrie," in Wolfgang U. Eckart and Klaus Volkert, ed., *Hermann von Helmholtz: Vorträge eines Heidelberger Symposiums anlässlich des einhundersten Todestages* (Pfaffenweiler: Centaurus-Verlagsgesellschaft, 1996), pp. 177–207.

[67] The non-Euclidean character of color space was emphasized by Erwin Schrödinger, "Outline of a Theory of Color Measurement for Daytime Vision" [1920] and "Thresholds of Color Differences" [1926], in MacAdam, *Sources of Color Science* (ref. 10), pp. 134–182, 183–193.

[68] Helmholtz, "Ueber die tatsächlichen Grundlagen" (ref. 59), pp. 616–617; "On the Factual Foundations," (ref. 59), pp. 50–51. Helmholtz goes on to cite his own work on eye movements from the first edition of his *Handbuch*. See also Gerhard Heinzmann, "The Foundations of Geometry and the Concept of Motion: Helmholtz and Poincaré," *Sci. in Con.* **14** (2001), 457–470.

[69] The second edition of Helmholtz's *Handbuch* mentions Riemann and describes color perception as a three-dimensional manifold comparable to space; H. von Helmholtz, *Handbuch der physiologischen Optik*. Zweite umgearbeitete Aufllage (Hamburg und Leipzig: Leopold Voss, 1896), p. 336.

[70] Helmholtz, *Sensations of Tone* (ref. 25), p. 370. Because this English version is based on the fourth German edition (1877), the reader should compare it with the second and third editions, Helmholtz, *Die Lehre von den Tonempfindungen* (ref. 31), p. 560; *idem*, Dritte umgearbeitete Auflage (Braunschweig: Friedrich Vieweg und Sohn, 1870), p. 576. As Vogel notes, Helmholtz's "exposition of the theory of the tone quality of musical instruments was essentially grounded in mathematics"; see Vogel, "Sensations of Tone," (ref. 21), p. 273.

[71] Helmholtz, *Sensations of Tone* (ref. 25), p. 370.

[72] These larger concerns also can be felt in his inaugural address as pro-rector at Heidelberg University; see Hermann von Helmholtz, "Ueber das Verhältnis der Naturwissenschaften zur Gesammtheit der Wissenschaftern" [1862], in *Vorträge und Reden*. Fünfte Auflage. Erster Band (ref. 7), pp, 157–185; translated as "On the Relation of Natural Science to Science," in Cahan, *Science and Culture* (ref. 62), pp. 76–95.

[73] Hermann Helmholtz, "Ueber den Ursprung und die Beudeutung der geometrischen Axioms" [1870], in *Vorträge und Reden*. Fünfte Auflage. Zweiter Band (ref. 19), pp. 1–31, on pp. 4–5; translated by E. Atkinson as "On the Origin and Significance of Geometrical Axioms," in *Popular Scientific Lectures* (ref. 7), pp. 223–249, on p. 224; and as "On the Origin and Meaning of Geometrical Axioms," in Pesic, *Beyond Geometry* (ref. 35), pp. 53–70, on p. 53. My translation here differs from Atkinson's, and elsewhere I sometimes also altered Atkinson's translation of *Mannigfaltigkeit* as "aggregate" to the now standard term "manifold."

[74] *Ibid.*, p. 25; 241; 64.

[75] *Ibid.*, p. 27; 243; 65–66.

[76] *Ibid.*, p. 19; 236; 61.

[77] *Ibid.*, p. 16; 233; 59.

[78] Though Kant had referred to space as a manifold, he had drawn back from using that term about time, guardedly noting that "we represent the time-sequence by a line progressing to infinity, in which the manifold constitutes a series of one dimension only"; see Immanuel Kant, *Critique of pure reason*. Translated and Edited by Paul Guyer and Allen W. Wood (Cambridge: Cambridge University Press, 1998), B50, p. 180. The word "manifold" may have ascribed more reality to time than Kant felt appropriate for it as "a purely subjective condition of our [human] intuition," thus "not something which exists of itself"; see *ibid.*, B51, B49, pp. 181, 180. Timothy Lenoir, "Operationalizing Kant: Manifolds, Models, and Mathematics in Helmholtz's Theories of Perception," in Michael Friedman and Alfred Nordmann, ed., *The Kantian Legacy in Nineteenth-Century Science* (Cambridge, Mass. and London: The MIT Press, 2006), pp. 141–210.

[79] Helmholtz, "Ueber den Ursprung" (ref. 73), p. 17; "On the Origin" (ref. 73), p. 234; "On the Origin and Meaning" (ref. 73), p. 59.

[80] In 1878 Helmholtz began to explore the factors he calls "topogenous" and "hylogenous" leading to space and time perceptions; see Helmholtz, "Thatsachen" (ref. 50), pp. 402–403; [Hertz and Schlick,] "Tatsachen" (ref. 50), pp. 149–150; [Kahl,] "Facts" (ref. 50), p. 405; [Cohen and Elkana,] "Facts" (ref. 50), pp. 159–160. David Jalal Hyder, "Helmholtz's Naturalized Conception of Geometry and his Spatial Theory of Signs," *Philosophy of Science* **66** (September 1999), S273–S286.

[81] Helmholtz, "On the Origin" (ref. 73), pp. 246–247; "On the Origin and Meaning" (ref. 73), p. 68.

[82] *Ibid.*, p. 246; 68, See also, for instance, S.P. Fullinwider, "Hermann von Helmholtz: The Problem of Kantian Influence," *Stud. Hist. Phil. Sci.* **21** (1990), 41–55.

[83] Helmholtz, "Ueber den Ursprung" (ref. 73), p. 8; "On the Origin" (ref. 73), p. 227; "On the Origin and Meaning" (ref. 73), p. 54. According to Reichenbach, "Helmholtz was the first to advocate the idea that human beings, living in a non-Euclidean world, would develop an ability of visualization which would make them regard the laws of non-Euclidean geometry as necessary and self-evident, in the same fashion as the laws of Euclidean geometry appear self-evident to us"; see Hans Reichenbach, "The Philosophical Significance of the Theory of Relativity," in Paul Arthur Schipp, ed., *Albert Einstein: Philosopher-Scientist* (Evanston, Ill.: The Library of Living Philosophers, 1949), pp. 289–311, on p. 308.

[84] Thomas Hawkins, "The Birth of Lie's Theory of Groups," *The Mathematical Intelligencer* **16** (1994), 6–17; *idem*, *The Emergence of the Theory of Lie Groups: An Essay in the History of Mathematics, 1869–1926* (New York: Springer-Verlag, 2000), pp. 124–130.

[85] William Kingdon Clifford, "The Postulates of the Science of Space" [1873], in Pesic, *Beyond Geometry* (ref. 35), pp. 73–87.

[86] Hawkins, *Emergence of the Theory of Lie Groups* (ref. 84), pp. 34–42; David E. Rowe, "Klein, Lie, and the 'Erlanger Programm'," in L. Boi, D. Flament, and J.-M. Salanskis, ed., *1830–1930: A Century of Geometry: Epistemology, History and Mathematics* (Berlin, Heidelberg: Springer-Verlag, 1992), pp. 45–54.

[87] Felix Klein, "The Most Recent Researches in Non-Euclidean Geometry" [1893], in Pesic, *Beyond Geometry* (ref. 35), pp. 109–116, on p. 110.

[88] Gerhard Heinzmann, "Helmholtz and Poincaré's Considerations on the Genesis of Geometry," in Boi, Flament, and Salanskis, *1830–1930: A Century of Geometry* (ref. 86), pp. 245–249; Heinzmann, "Foundations of Geometry" (ref. 68).

[89] Henri Poincaré, "Non-Euclidean Geometries" [1891], in Pesic, *Beyond Geometry* (ref. 35), pp. 97–105, on p. 100. For discussion of the Poincaré and Klein models, see Rosenfeld, *History of Non-Euclidean Geometry* (ref. 61), pp. 236–246.

[90] Howard Stein, "Some Philosophical Prehistory of General Relativity," in John Earman, Clark Glymour, and John Stachel, ed., *Minnesota Studies in the Philosophy of Science*. Vol. 8.

Foundations of Space-Time Theories (Minneapolis: University of Minnesota Press, 1977), pp. 3–49, on pp. 21–25; Martin Carrier, "Geometric Facts and Geometric Theory: Helmholtz and 20th-Century Philosophy of Physical Geometry," in Krüger, *Universalgenie Helmholtz* (ref. 17), pp. 276–291; Michael Friedman, "Geometry as a Branch of Physics: Background and Context for Einstein's 'Geometry and Experience'," in David B. Malament, ed., *Reading Natural Philosophy: Essays in the History and Philosophy of Science and Mathematics* (Chicago and La Salle: Open Court, 2002), pp. 193–229; Josć Ferreirós, "Riemann's Habilitationsvortrag at the Crossroads of Mathematics, Physics, and Philosophy," in J. Ferreirós and J. Gray, ed., *The Architecture of Modern Mathematics: Essays in History and Philosophy* (Oxford and New York: Oxford University Press, 2006), pp. 67–96.

[91] Einstein to Thornton, December 7, 1944, quoted in Don A. Howard, "Albert Einstein as a Philosopher of Science," *Physics Today* **58** (December 2005), 34–40, on 34.

[92] Albert Einstein, "The Problem of Space, Ether, and the Field in Physics" [1934], in Pesic, *Beyond Geometry* (ref. 35), pp. 187–193, on p. 190.

[93] Riemann did not include these details in his 1854 lecture but provided them in his "Commentatio mathematica" [1861], in *Gesammelte Mathematische Werke* (ref. 35), pp. 423–436. Ruth Farwell and Christopher Knee, "The Missing Link: Riemann's 'Commentatio,' Differential Geometry, and Tensor Analysis," *Hist. Math.* **17** (1990), 223–255.

[94] Albert Einstein and Leopold Infeld, *The Evolution of Physics: The Growth of Ideas from Early Concepts to Relativity and Quanta* (New York: Simon and Schuster, 1936), p. 33. For Einstein's reading of Riemann, see ref. 2 above; though Einstein surely knew Riemann's 1854 lecture, it is not clear whether he had read his work on the mechanism of the ear.

[95] Einstein's review of a wartime (1917) edition of Helmholtz's, *Zwei Vorträge über Goethe*, in A.J. Kox, Martin J. Klein, and Robert Schulmann, ed., *The Collected Papers of Albert Einstein. Vol. 6. The Berlin Years: Writings 1914–1917* (Princeton: Princeton University Press, 1996), p. 569.

[96] Albert Einstein, "Non-Euclidean Geometry and Physics" [1925], in Pesic, *Beyond Geometry* (ref. 35), pp. 159–162, on p. 161.

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